

## Spline Semiparametric Regression Models

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### ABSTRACT

In this paper, we study semiparametric regression models with spline smoothing, and determining the numbers of knots and their locations by using some statistical criteria, a simulation model has been performed.

### Keywords

Spline, Penalized spline, Mixed Models, Semiparametric, Regression, knots, cross-validation.

### 1. Introduction

Regression analysis is a statistical tool that utilizes the relation between two or more quantitative variables so that one variable can be predicted from the other, or others. For example, if one knows the relation between advertising expenditures and sales, one can predict sales by regression analysis once the level of advertising expenditures has been set.

Linear regression is a statistical modeling technique that relates the change in one variable to other variables ( see [9]).

A simple linear regression line has an equation of the form  $y = \beta_0 + \beta_1 x + \varepsilon$ , where  $x$  is the explanatory variable and  $y$  is the dependent variable. The slop of the line is  $\beta_1$ ,  $\beta_0$  is the intercept, and  $\varepsilon$  is an error term ( see [11]).

In many applications in different fields, we need to use one of a collection of models for correlated data structures, for example, multivariate observations clustered data, repeated measurements, longitudinal data and spatially data. Often random effects are used to describe the correlation structure in this type of this data. Mixed models are an extension of regression models that allow for the incorporation of random effects. However, they also turn out to be closely related to smoothing ( see [13]).

In this paper we study semiparametric regression models with spline smoothing, and we present the definition, properties of the statistical models, estimation method. Also we present the number of knots and their locations and the simulation models has been performed. The sample size taken are  $n = 20, 100$  and  $250$ .

### 2. Nonparametric regression

Given data of the form  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ . Let the model ( see [5]):

$$y = g(x) + \varepsilon \quad (1)$$

Where the noise term  $\varepsilon$  satisfies the usual conditions assumed for simple linear regression, we seek an estimate of the regression function  $g(x)$  satisfying the model (1). There are several approaches to this problem, we will describe methods involving splines.

### 3. Splines

The discovery that piecewise polynomials or splines could be used in place of polynomials occurred in the early twentieth century. Splines have since become one of the most popular ways of approximating nonlinear functions. Splines are essentially defined as piecewise. Let  $k$  be any real number, then can define a  $p^{th}$  degree truncated power function as ( see [2,3,4,7,8]):

$$(x - k)_+^p = (x - k)^p I_{\{x > k\}}(x) \quad (2)$$

As a function of  $x$ , this function takes on the value 0 to the left of  $k$ , and it takes on the value  $(x - k)^p$  to the right of  $k$ . The number  $K$  is called a knot.

The above truncated power function is a basic example of a spline. It is a member of the set of basis functions for the space of splines.

Let us consider a general  $p^{th}$  degree spline with a single knot at  $k$ . Let  $P(x)$  denote an arbitrary  $p^{th}$  degree polynomial.

$$P(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \dots + \beta_p x^p$$

Then :

$$S(x) = P(x) + \beta_{p+1}(x - k)_+^p \quad (3)$$

takes on the value  $P(x)$  for any  $x \leq k$ , and it takes on the value

$$P(x) + \beta_{p+1}(x - k)^p \text{ for any } x > k$$

Thus, restricted to each region, the function is a  $p^{th}$  degree polynomial. As a whole, this function is a  $p^{th}$  degree piecewise polynomial; there are two pieces.

Note that require  $p + 2$  coefficients to specify this piecewise polynomial. This is a result of the addition of the truncated power function specified by the knot at  $k$ . In general, we may add  $K$  truncated power function specified at  $k_1, k_2, \dots, k_K$ , each multiplied by different coefficients. Thus would result in  $p+K+1$  degree of freedom.

An important property of splines is their smoothness. Polynomials are very smooth, possessing all derivatives everywhere. Splines possess all derivatives only at points which are not knots. The number of derivatives at a knot depends on the degree of the spline, consider the spline by (3), we can show that  $S(x)$  is continuous at  $k$ , when  $p > 0$  by noting that:

$$S(k) = P(k)$$

$$\text{and } \lim_{x \rightarrow k} \beta_{p+1}(x - k)_+^p = 0$$

so that  $\lim_{x \rightarrow k} S(x) = P(k)$   
can argue similarly for the first  $p - 1$  derivatives

$$S^{(j)}(k) = P^{(j)}(k), \quad j = 1, 2, \dots, p-1$$

And

$$\lim_{x \rightarrow k} \beta_{p+1} p(p-1) \dots (p-j+1) (x-k)^{p-j} = 0$$

So that

$$\lim_{x \rightarrow k} S^{(j)}(x) = P^{(j)}(k)$$

The  $p^{th}$  derivative behaves differently :

$$S^{(p)}(t) = p! \beta_p$$

$$\text{and } \lim_{x \rightarrow k} S^{(p)}(x) = p! \beta_p + p! \beta_{p+1}$$

So usually there is a discontinuity in the  $p^{th}$  derivative. Thus  $p^{th}$  degree splines are usually said to have no more than  $(p-1)$  continuous derivatives.

The discussion below (3) indicates that can represent any piecewise polynomials of degree  $p$  in the following way:

$$S(x) = \beta_0 + \beta_1 x + \dots + \beta_p x^p + \beta_{p+1}(x - k_1)_+^p + \dots + \beta_{p+K}(x - k_K)_+^p \quad (4)$$

Any piecewise polynomial can be expressed as a linear combination of truncated power functions and polynomial of degree  $p$

$$S(x) = \begin{cases} \beta_0 + \beta_1 x + \dots + \beta_p x^p, & x \leq k_1 \\ \beta_0 + \beta_1 x + \dots + \beta_p x^p + \beta_{p+1}(x - k_1)^p, & k_1 < x \leq k_2 \\ \beta_0 + \beta_1 x + \dots + \beta_p x^p + \beta_{p+1}(x - k_1)^p + \beta_{p+2}(x - k_2)^p, & k_2 < x \leq k_3 \\ \vdots \\ \beta_0 + \beta_1 x + \dots + \beta_p x^p + \beta_{p+1}(x - k_1)^p + \dots + \beta_{p+K}(x - k_K)^p, & x > k_K \end{cases}$$

In the other words,

$$\{1, x, x^2, \dots, x^p, (x - k_1)_+^p, (x - k_2)_+^p, \dots, (x - k_K)_+^p\}$$

Is a basis for the space of  $p^{th}$  degree splines possessing knots at  $k_1, k_2, \dots, k_K$ . By adding a noise term to (4), we can obtain a splines regression model relating a response

$$Y = S(x) + \varepsilon \quad (5)$$

to the predictor  $x$ .

### 3.1 Penalized splines

Let us consider the model (1) with linear spline ( knots ) as ( see [1,12]):

$$S(x) = \beta_0 + \beta_1 x + \sum_{j=1}^q \beta_{1j} (x - k_j)_+$$

then the ordinary least squares fit can be written:

$$\hat{Y} = X \hat{\beta},$$

where  $\hat{\beta}$  minimizes  $\|Y - X\beta\|^2$ , with  $\beta = (\beta_0, \beta_1, \beta_{11}, \beta_{12}, \dots, \beta_{1q})^T$  and with

$$X = \begin{bmatrix} 1 & x_1 & (x_1 - k_1)_+ & \dots & (x_1 - k_q)_+ \\ 1 & x_2 & (x_2 - k_1)_+ & \dots & (x_2 - k_q)_+ \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & (x_n - k_1)_+ & \dots & (x_n - k_q)_+ \end{bmatrix}$$

Unconstrained estimation of  $\beta_{11}, \beta_{12}, \dots, \beta_{1q}$  leads to a wiggly fit. For judicious choice of  $C$ , a constraint of the type:

$$\sum_{j=1}^q \beta_{1j}^2 < C$$

If we define the  $(q+2) \times (q+2)$  matrix.

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix} = \begin{bmatrix} 0_{2 \times 2} & 0_{2 \times q} \\ 0_{q \times 2} & I_{q \times q} \end{bmatrix}$$

The minimization problem can be written as:

$$\text{Min } \|Y - X\beta\|^2$$

Subject to

$$\beta^T D \beta < C$$

It can be shown, using a Lagrange multiplier argument, that this is equivalent to choosing  $\beta$  to minimize :

$$\|Y - X\beta\|^2 + \lambda^2 \beta^T D \beta \quad (6)$$

for some  $\lambda \geq 0$ . This has the solution

$$\hat{\beta}_\lambda = (X^T X + \lambda^2 D)^{-1} X^T Y \quad (7)$$

The term  $\lambda^2 \beta^T D \beta$  is called a roughness penalty because it penalizes fits that are too rough, thus yielding a smoother result. The amount of smoothing is controlled by  $\lambda$  ( the smoothing parameter ).

When the value of the smoothing parameter ( $\lambda$ ) is very large then  $\beta_{1k} \rightarrow 0$  leads to the estimator is polynomials of degree  $q$  only, while if the  $\lambda = 0$  then leads to no exist roughness penalty.

### 3.2 Number and position of knots

If the number of knots too small, then the bias can be large in estimator, and if the number too large it is, preferred, we can use all the observations as knots.

Literature proposes several a approaches to automatic knot selection. Many of them are based on stepwise regression ideas. Although most of the automatic knot selection procedures proposed exhibit good performance they are each quite complicated and computationally intensive. In penalized spline the number of knots ( $K$ ) that usually works well is :

$$K = \min \left( \frac{1}{4} \text{ number of unique } x_i, 35 \right), \quad (\text{see [8,12,15]}):$$

As the position of knots determine from the  $\left( \frac{k+1}{K+2} \right)^{th}$  sample quantile of the unique  $x_i$  for  $k = 1, 2, \dots, K$ .

### 3.3 Cross Validation (CV)

Let  $\hat{m}(x, \lambda)$  denote the regression estimate at a point  $x$  with smoothing parameter  $\lambda$ . One of the most common measures for the goodness of fit of a regression curve to a scatter plot is the residual sum of squares ( $RSS$ ):

$$RSS(\lambda) = \sum_{i=1}^n (y_i - \hat{y}_i)^2 \quad (8)$$

With  $\hat{y}_i = \hat{m}(x_i, \lambda)$ . However, since  $RSS$  is minimized at the interpolant ( $\hat{y}_i = y_i$ ,  $i = 1, 2, \dots, n$ ), minimization of this criterion will lead to the smooth that is closest to interpolation. For penalized spline this corresponds to a zero smoothing parameter. Cross validation gets around this problem. The cross validation criterion is ( see [5,12]):

$$CV(\lambda) = \sum_{i=1}^n [y_i - \hat{m}_{-i}(x_i, \lambda)]^2 \quad (9)$$

Where  $\hat{m}_{-i}$  denotes the regression estimator applied to the data but with  $(x_i, y_i)$  deleted. The

CV choice of  $\lambda, \hat{\lambda}_{CV}$  is the one that minimizes  $CV(\lambda)$  over  $\lambda \geq 0$ .

### 3.4 Generalized Cross Validation ( GCV )

Efficient algorithms for computation of  $CV(\lambda)$  were developed in the mid1980s. Before that time, the difficulties surrounding computation of the cross-validation criterion led to the proposal of a simplified version. This simplified criterion is known a generalized cross-validation.

$$GCV(\lambda) = \frac{n \sum_{i=1}^n ((I - S_\lambda) y)_i^2}{(tr(I - S_\lambda))^2} = \frac{n \sum_{i=1}^n (y_i - \hat{m}(x_i, \lambda))^2}{(tr(I - S_\lambda))^2} = \frac{n RSS(\lambda)}{(tr(I - S_\lambda))^2} \quad (10)$$

where  $S_\lambda$  be the smoother matrix associated with  $\hat{m}$  and  $S_\lambda$  satisfy  $\hat{Y} = S_\lambda Y$  ( see [5,12]).

### 4-Mixed Models

Mixed models are an extension of regression models that allow for the incorporation of random effects. A more contemporary application of mixed models is the analysis of longitudinal data, clustered data repeated measurements and spatially correlated data. The general form of a linear mixed model is given as follows ( see [12]):

$$Y_i = X_i \beta + \sum_{j=1}^r Z_{ij} u_{ij} + \epsilon_i \quad (11)$$

$$u_{ij} \sim N(0, G_j), \quad \epsilon_i \sim N(0, R_i)$$

where the vector  $Y_i$  has length  $m_i$ ,  $X_i$  and  $Z_{ij}$  are, respectively, a  $m_i \times p$  design matrix and a  $m_i \times q_i$  design matrix of fixed and random effects.  $\beta$  is a  $p$ -vector of fixed effects and  $u_{ij}$  are the  $q_i$ -vectors of random effects. The variance matrix  $G_j$  is a  $q_i \times q_i$  matrix and  $R_i$  is a  $m_i \times m_i$  matrix.

We assume that the random effects  $\{u_{ij}; i = 1, \dots, n; j = 1, \dots, r\}$  and the set of error terms  $\{\epsilon_1, \dots, \epsilon_n\}$  are independent. In matrix notation,

$$Y = X\beta + Zu + \epsilon \quad (12)$$

Here  $Y = (Y_1, \dots, Y_n)^T$  has length  $N = \sum_{i=1}^n m_i$ ,  $X = (X_1^T, \dots, X_n^T)^T$  is a  $N \times p$  design matrix of fixed effects,  $Z$  is a  $N \times q$  block diagonal design matrix of random effects,  $q = \sum_{j=1}^r q_j$ ,  $u = (u_1^T, \dots, u_r^T)^T$  is a  $q$ -vector of random effects,  $R = \text{diag}(R_1, \dots, R_n)$  is a  $N \times N$  matrix and  $G = \text{diag}(G_1, \dots, G_r)$  is a  $q \times q$  block diagonal matrix.

We now treat estimation of  $\beta$ , prediction of  $u$ , and estimation of the parameters in  $G$  and  $R$ , one way to drive an estimate of  $\beta$  is to rewrite (12) as:

$$Y = X\beta + \varepsilon^*, \text{ where } \varepsilon^* = Zu + \varepsilon$$

This is just a linear model with correlated error, since:

$$\text{cov}(\varepsilon^*) \equiv V = ZGZ^T + R$$

For given  $V$ , the estimator of  $\beta$  is:

$$\tilde{\beta} = (X^T V^{-1} X)^{-1} X^T V^{-1} Y \quad (13)$$

And is sometime referred to as generalized linear squares (GLS). For  $Y$  having a general distribution (13) can be shown to be the best linear unbiased estimator (BLUE) for  $\beta$ . Alternatively, if  $Y$  is multivariate normal, then the right hand side of (13) is both the maximum likelihood estimator (MLE) and the uniformly minimum variance unbiased estimator (UMVUE).

The latter is the estimator that has the best (smallest) possible variance of any unbiased estimator regardless of the parameters values [12].

The random effects vector can be predicted via best linear prediction.

$$\tilde{u} = \text{BLP}(u) = GZ^T V^{-1} (Y - X\tilde{\beta}) \quad (14)$$

Then the BLUP of  $(\beta, u)$  can also be written as:-

$$\begin{bmatrix} \tilde{\beta} \\ \tilde{u} \end{bmatrix} = (C^T R^{-1} C + B)^{-1} C^T R^{-1} Y \quad (15)$$

Where

$$C \equiv [X \ Z] \text{ and } B = \begin{bmatrix} 0 & 0 \\ 0 & G^{-1} \end{bmatrix}$$

The fitted values are then:

$$\text{BLUP}(Y) = X\tilde{\beta} + Z\tilde{u} = C(C^T R^{-1} C + B)^{-1} C^T R^{-1} Y = HY \quad (16)$$

where  $H$  called Hat matrix or smoother matrix,

The Log - likelihood of  $Y$  under the model  $Y \sim N(X\beta, V)$  is:-

$$L(\beta, V) = -\frac{1}{2} \{ n \log(2\pi) + \log|V| + (Y - X\beta)^T V^{-1} (Y - X\beta) \} \quad (17)$$

By substitution (13) in (17) we obtain the profile log - likelihood for  $V$ :

$$\begin{aligned} L_p(V) &= -\frac{1}{2} \{ \log|V| + (Y - X\tilde{\beta})^T V^{-1} (Y - X\tilde{\beta}) + n \log(2\pi) \} \\ &= -\frac{1}{2} \{ \log|V| + Y^T V^{-1} [I - X(X^T V^{-1} X)^{-1} X^T V^{-1}] Y \} - \frac{n}{2} \log(2\pi) \end{aligned} \quad (18)$$

#### 4-1 Penalized spline as BLUPs

The penalized spline fitting criterion (6), when divided by  $\sigma_\varepsilon^2$  can then be written as ( see [12]):

$$\frac{1}{\sigma_\varepsilon^2} \|Y - X\beta - Zu\|^2 + \frac{\lambda^2}{\sigma_\varepsilon^2} \|u\|^2 \quad (19)$$

Notice that this can be made to equal the BLUP criterion by treating the  $u$  as a set of random coefficients with:

$$\text{cov}(u) = \sigma_u^2 I, \quad \text{where } \sigma_u^2 = \frac{\sigma_\varepsilon^2}{\lambda^2}$$

Putting all of this together yields the mixed model representation of the regression spline

$$Y = X\beta + Zu + \varepsilon \text{ and } f = X\beta + Zu$$

$$\text{cov} \begin{bmatrix} u \\ \varepsilon \end{bmatrix} = \begin{bmatrix} \sigma_u^2 I & 0 \\ 0 & \sigma_\varepsilon^2 I \end{bmatrix} \quad (20)$$

note that the fitted values  $\tilde{f}$  can be rewritten as:

$$\tilde{f} = C(C^T C + \lambda^2 D)^{-1} C^T Y \quad (21)$$

#### 5- Semiparametric Models

Let the model:-

$$Y_i = \sum_{j=0}^p \beta_j X_{ji} + m(X_{p+1,i}) + \varepsilon, \quad i = 1, \dots, n \quad (22)$$

We call (22) the Semiparametric models it has a nonparametric component  $m(X_{p+1,i})$  and parametric component  $\sum_{j=0}^p \beta_j X_{ji}$ . By using penalized spline of degree  $q$ , get:

$$y_i = \sum_{j=0}^p \beta_j x_{ji} + \sum_{j=1}^q \beta_{p+j} x_{p+1,i}^j + \sum_{k=1}^K \{u_k(x_{p+1,i} - k_k)_+^q + \varepsilon_i\} \quad (23)$$

where  $k_1, \dots, k_K$  are inner knots

$$a < k_1 < \dots < k_K < b.$$

By using a convenient connection between penalized splines and mixed models. Model (23) is rewritten as follows ( see [6,10,12,13,14])

$$Y = X\beta + Zu + \varepsilon \quad (24)$$

where

$$Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_0 \\ \vdots \\ \beta_p \\ \beta_{p+1} \\ \vdots \\ \beta_{p+q} \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ \vdots \\ u_K \end{bmatrix}$$

$$Z = \begin{bmatrix} (x_{p+1,1} - k_1)_+^q & \dots & (x_{p+1,1} - k_K)_+^q \\ \vdots & \ddots & \vdots \\ (x_{p+1,n} - k_1)_+^q & \dots & (x_{p+1,n} - k_K)_+^q \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & x_{11} & \dots & x_{p1} & x_{p+1,1} & \dots & x_{p+1,1}^q \\ 1 & x_{12} & \dots & x_{p2} & x_{p+1,2} & \dots & x_{p+1,2}^q \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{1n} & \dots & x_{pn} & x_{p+1,n} & \dots & x_{p+1,n}^q \end{bmatrix}$$

Assume that  $u$  and  $\varepsilon$  are independent and normally distributed as  $u \sim N(0, G)$ ,  $\varepsilon \sim N(0, R)$ ,

The estimation of the parameters  $\beta$  and  $u$  entails minimizing the penalized least squares criterion

$$\|Y - X\beta - Zu\|^2 + \lambda^2 u^T D u; \quad (25)$$

Where  $D$  is penalty matrix. For a given smoothing parameter matrix  $D$ , the penalized least squares estimators from (25) are :

$$\begin{pmatrix} \hat{\beta} \\ \hat{u} \end{pmatrix} = \begin{pmatrix} X^T X & X^T Z \\ Z^T X & Z^T Z + D \end{pmatrix}^{-1} \begin{pmatrix} X^T \\ Z^T \end{pmatrix} Y \quad (26)$$

and the fitted values are  $\hat{Y} = X\hat{\beta} + Z\hat{u} = HY$ , where  $H$  is the smoothing matrix given by

$$H = (X \ Z) \begin{pmatrix} X^T X & X^T Z \\ Z^T X & Z^T Z + D \end{pmatrix}^{-1} \begin{pmatrix} X^T \\ Z^T \end{pmatrix} \quad (27)$$

Where  $H$  is smoothing matrix.

## 6. Numerical results:

The settings for the simulation study are as follows. The observations for the design variable  $x_1, \dots, x_n$  are generated from uniform distribution on the interval  $[-1, 1]$ , for various sample sizes. These values are kept fixed for all settings to reduce simulation variability. The sample sizes taken are  $n = 20, 100$  and  $250$ .

We have used the following two semiparametric regression functions, which represent a variety of shapes,

$$m_1(x) = e^{\cos(\pi x + 2x)} - x,$$

$$m_2(x) = \sin(2\pi x) + 0.3(x - 0.75)^2.$$

For the error distribution we used normal distribution  $N(0, \sigma_\varepsilon^2)$ , where  $\sigma = 0.125, 0.25$  and  $0.5$ . We have tried with different choices of  $K$  as well. The penalty parameter  $\lambda$  is chosen by minimizing the generalized cross validation (GCV) criterion, results shown in table (1).

To give an impression on the variability of the obtained estimators, we plot in figure 1-6 a scatter plot of the randomly generated data sets together with the fitted values from the penalized LS. regression spline estimation method. The goodness of fit of the estimated models quantified by computing the following criterions:

1) Mean Squared Error ( $MSE$ ) is defined by:

$$MSE = \frac{1}{n} \sum_{i=1}^N [m(x_i) - \hat{m}(x_i)]^2$$

2) Mean Absolute Error ( $MAE$ ) is defined by:

$$MAE = \frac{1}{n} \sum_{i=1}^N |m(x_i) - \hat{m}(x_i)|$$

3) Average Mean Squared Error ( $AMSE$ ) is defined by:

$$AMSE = \frac{1}{N} \sum_{i=1}^N MSE(x_i)$$

4) Average Mean Absolute Error ( $AMAE$ ) is defined by:

$$AMAE = \frac{1}{N} \sum_{i=1}^N MAE(x_i)$$

Where  $N$  is a number of frequencies

Table (1) presents summary values of the ( $AMSE$ ) and ( $AMAE$ ) for the estimation methods. In all cases when the sample sizes ( $250$ ), the values of ( $AMSE$ ) and ( $AMAE$ ) is smaller than that of the other two size ( $20$ ) and ( $100$ ) so, we can see from the figures 1-6 the results is better with sample size ( $n=250$ ) for two test functions from table (1) we can see the values of ( $AMSE$ ) and ( $AMAE$ ) with sample ( $250$ ) ( $0.00002879090$ ) and ( $0.004246073$ ), with number of knots ( $34$ ) for the first test function, while for second test function the values of ( $AMSE$ ) and ( $AMAE$ ) was ( $0.00002989973$ ) and ( $0.004327188$ ) respectively with number of knots ( $34$ ).

N	function	$\sigma_\varepsilon$	size	AMSE	AMAE	df	spar	knots
1	$S(x) = e^{\cos(\pi x + 2x)} - x$	0.125	20	0.08305458000	0.180457200	5.912	0.09439	4
			100	0.00005491964	0.005828587	23.65	0.03517	24
			250	0.00002879090	0.004246073	31.44	0.04113	34
		0.25	20	0.0856969100	0.18608910	5.911	0.09483	4
			100	0.0006275509	0.01975109	18.98	0.07999	24
			250	0.0003134251	0.01397690	23.93	0.08889	34
		0.5	25	0.136985100	0.25500340	5.757	0.13980	4
			100	0.006863818	0.06545415	12.94	0.17330	24
			250	0.003228072	0.04463233	15.65	0.18840	34
2	$S(x) = \sin(2\pi x) + 0.3(x - 0.75)^2$	0.125	20	0.16142910000	0.287354800	4.683	0.46440	4
			100	0.00005968586	0.006073870	24.04	0.03153	24
			250	0.00002989973	0.004327188	31.94	0.03797	34
		0.25	20	0.1606705000	0.28676900	4.501	0.09443	4
			100	0.0006776400	0.02045947	19.68	0.07276	24
			250	0.0003282252	0.01429815	24.67	0.08343	34
		0.5	25	0.208411800	0.35126800	4.499	0.5395	4
			100	0.007002548	0.06578606	13.60	0.1586	24
			250	0.003317835	0.04514782	15.97	0.1824	34

Table (1) result of the simulation study

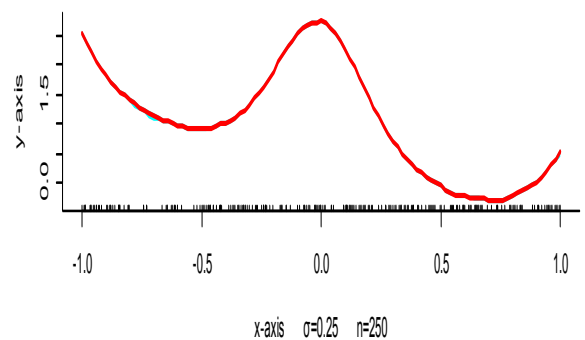
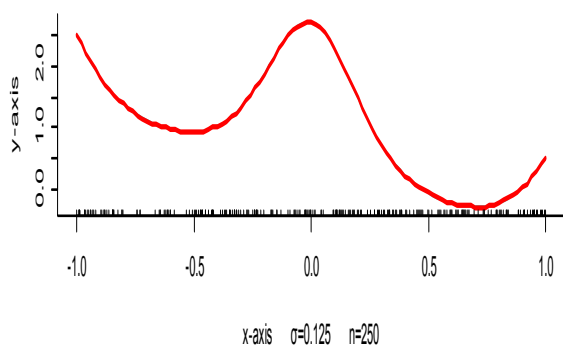
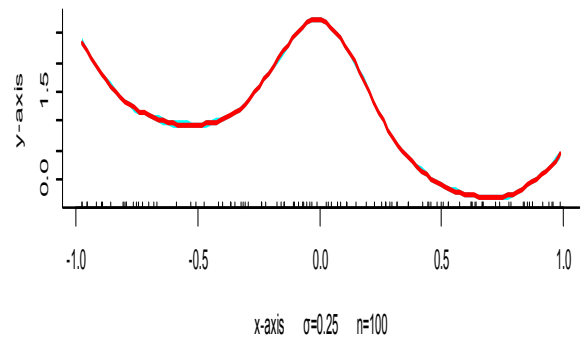
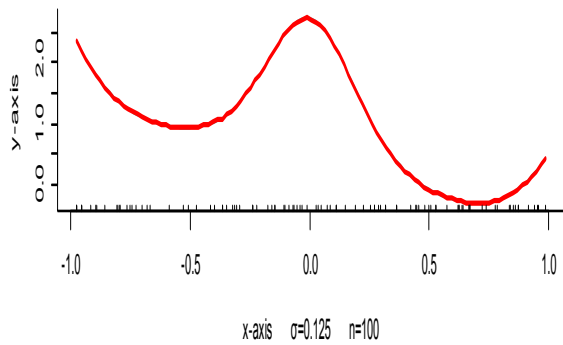
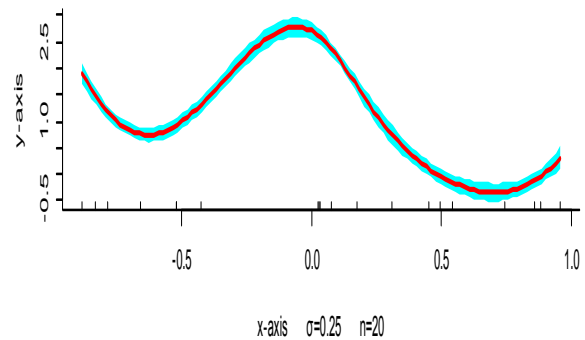
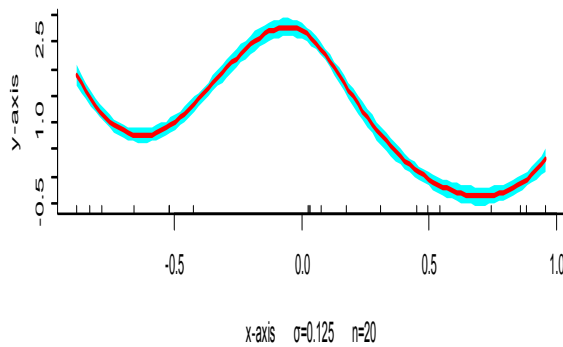


Figure 1 fitted curves from penalized regression spline estimation of first test function with design variable  $X$  distributed uniform distribution  $[-1,1]$  and the error distributed normal distribution  $(0, 0.125^2)$ , and sample size  $n=20,100$  and  $250$ .

Figure 2 fitted curves from penalized regression spline estimation of first test function with design variable  $X$  distributed uniform distribution  $[-1,1]$  and the error distributed normal distribution  $(0, 0.25^2)$ , and sample size  $n=20,100$  and  $250$ .

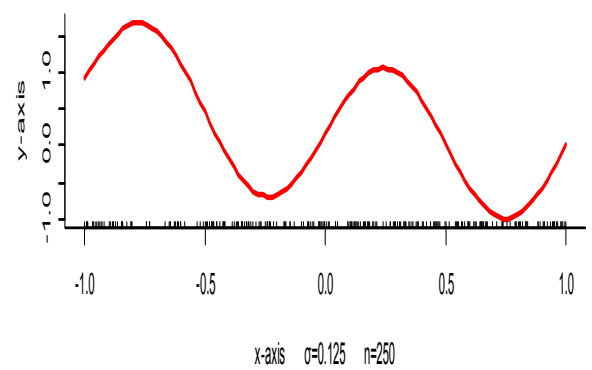
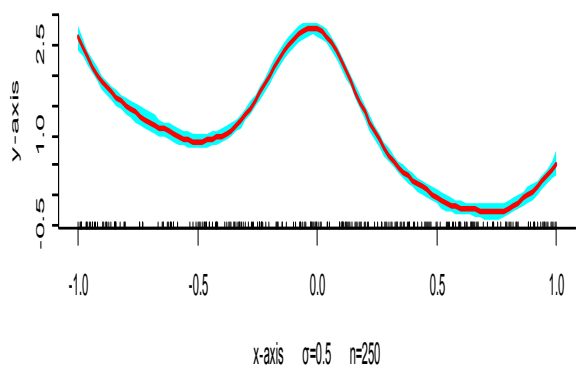
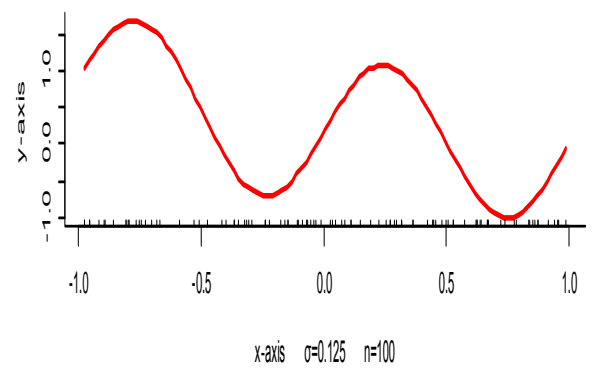
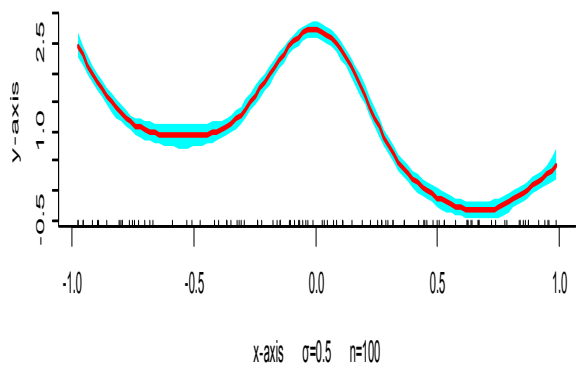
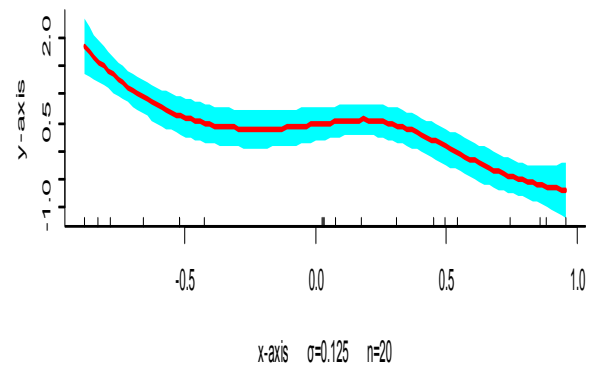
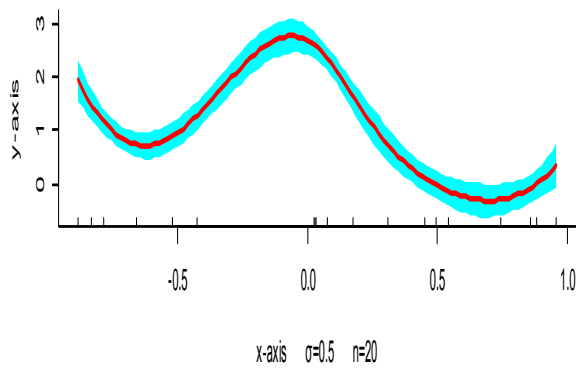
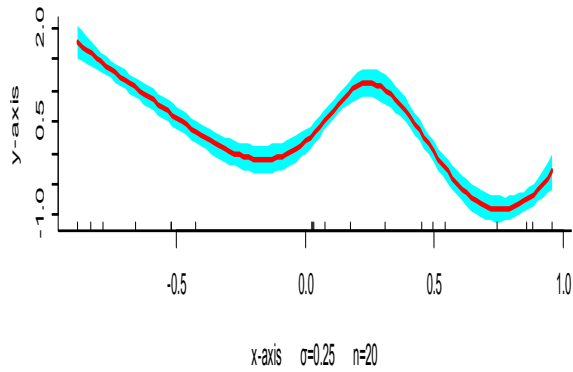


Figure 3 fitted curves from penalized regression spline estimation of first test function with design variable  $X$  distributed uniform distribution  $[-1,1]$  and the error distributed normal distribution  $(0, 0.5^2)$ , and sample size  $n=20,100$  and  $250$ .

Figure 4 fitted curves from penalized regression spline estimation of second test function with design variable  $X$  distributed uniform distribution  $[-1,1]$  and the error distributed normal



distribution  $(0, 0.125^2)$ , and sample size  $n=20,100$  and  $250$ .



design variable  $X$  distributed uniform distribution  $[-1,1]$  and the error distributed normal distribution  $(0, 0.25^2)$ , and sample size  $n=20,100$  and  $250$ .

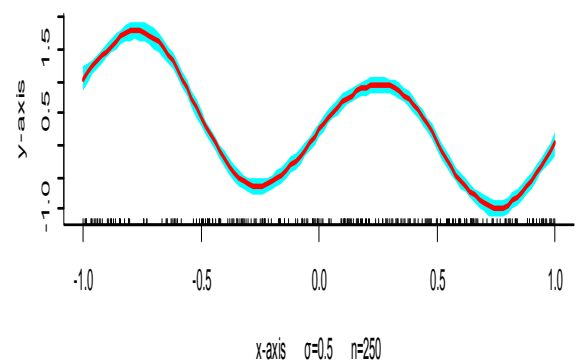
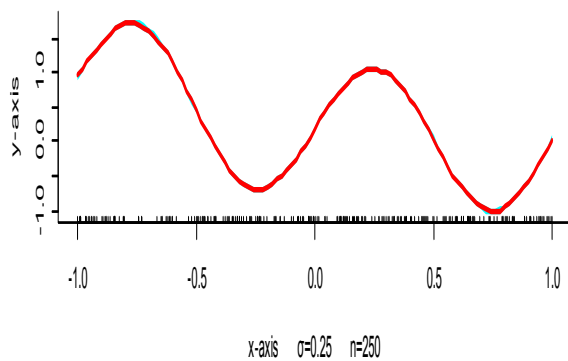
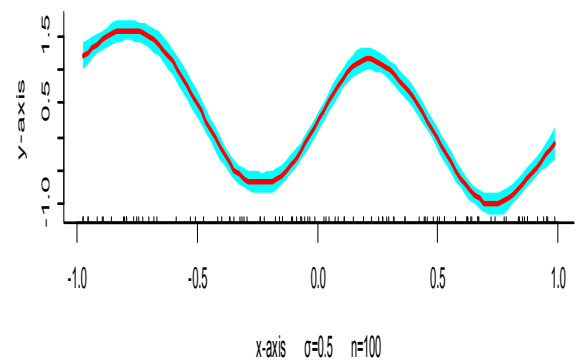
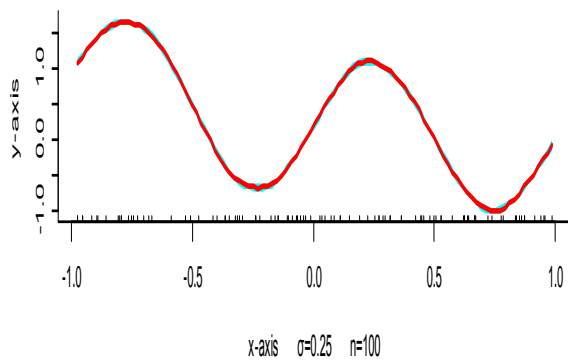
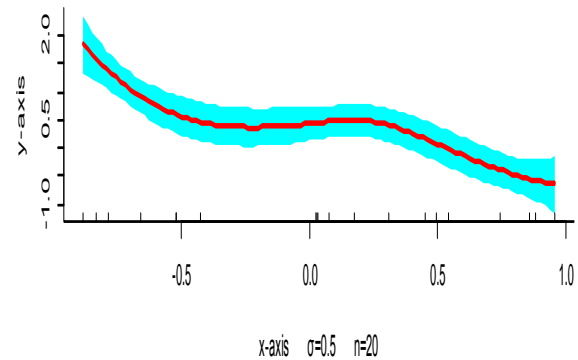


Figure 5 fitted curves from penalized regression spline estimation of second test function with

Figure 6 fitted curves from penalized regression spline estimation of second test function with design variable  $X$  distributed uniform distribution  $[-1,1]$  and the error distributed normal distribution  $(0, 0.5^2)$ , and sample size  $n=20,100$  and  $250$ .

## 7. Conclusion:

From the result for the test functions we can see With increase the value of sigma of the normal distribution getting decreases degree of freedom and increases in value of smoothing parameter.

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