BIFURCATION DIAGRAM FOR NONLINEAR SYSTEM OF ALGEBRAIC EQUATIONS WITH PARAMETERS

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Abstract. In this paper we are interested in studying of bifurcation solutions of bifurcation equation (nonlinear system of algebraic equations with four or six parameters). Also, we found a new geometrical description of the Discriminate set(bifurcation set) with bifurcation spreading of the number of regular solutions in every region. In addution, we calculate the topological indices of the solutions of the problem.

Key Words: Nonlinear system of algebraic equations, Bifurcation diagram, Regular solutions.

1. Introduction.

Equations with parameters are important to be studied in many physical and mathematical problems. The equations may be written as

$$\mathrm{L}(\mathrm{x},\lambda) = \mathrm{b}, \quad \mathrm{x} \in O \subset X, \ \mathrm{b} \in Y, \ \lambda \in \mathrm{R}^{\mathrm{n}}. \qquad \dots (1.1)$$

where L is a smooth Fredholm mapping of zero index between Banach spaces X and Y and O is an open subset in Y. The solutions of these problems can be found by solving the equivalent equation,

$$\Phi\big(\xi,\lambda\big) = \beta, \quad \xi \in M, \quad \beta \in N, \qquad \dots (1.2)$$

where M and N are smooth finite dimensional manifolds.

The reduction from equation (1.1) to the equation (1.2) (Variant local scheme of Lyapunov-Schmidt) with the conditions that equation (1.2) has all the topological and analytical properties of equation (1.1) (multiplicity, bifurcation diagram, *etc*) dealing with [4],[14],[16][17] .

In the method of finite dimensional reduction the solutions of infinite dimensional spaces coincide with the solutions of finite dimensional spaces, for this reason the method became important in modern mathematics. The system of nonlinear algebraic equations is important in physics , engineering, particularly in mechanical and electrical engineering. It is important in population growth, economy, and many other phenomena. In recent years, there are many

studies of different types of bifurcation solutions.

In M.A. Abdul Hussain [7], studied bifurcation periodic solutions of the following nonlinear system of algebraic equations,

$$\begin{aligned}
&\xi_1(\xi_1^2+\xi_2^2)+2\xi_1(\xi_3^2+\xi_4^2)+q_1(\xi_2\,\xi_3-\xi_1\,\xi_4)+q_2\xi_1=0\\
&\xi_2(\xi_1^2+\xi_2^2)+2\xi_2(\xi_3^2+\xi_4^2)-q_1(\xi_1\,\xi_3+\xi_2\,\xi_4)+q_2\xi_2=0\\
&2\xi_3(\xi_1^2+\xi_2^2)+\xi_3(\xi_3^2+\xi_4^2)+\alpha_1\xi_1\,\xi_2+\alpha_2\xi_3=0\\
&2\xi_4(\xi_1^2+\xi_2^2)+\xi_4(\xi_3^2+\xi_4^2)+\beta_1(\xi_1^2-\xi_2^2)+\alpha_2\,\xi_4=0
\end{aligned}$$

And taking the example of the following equation

$$u^{(4)}-\frac{12}{r^2}u^{//}+\frac{12}{k\,r}u\,u^{//}+\frac{6}{k\,r}(u^{/})^2-\frac{36(r-\frac{k^2}{r})}{k^2\,r^3}u$$
$$-\frac{54}{k\,r^3}u^2+\frac{18}{k^2\,r^2}u^3=0.$$

In M.A. Abdul Hussain [8], studied bifurcation solutions of the following nonlinear system of algebraic equations,

$$\begin{aligned}
&\xi_1^3+2\xi_1\,\xi_2^2+\lambda_1\,\xi_1+q_1=0\\
&\xi_2^3+2\xi_1^2\,\xi_2+\lambda_2\,\xi_2+q_2=0.
\end{aligned}$$

And taking the example of the following equation without symmetry of $\psi(x)$

$$\frac{d^4u}{d\,x^4}+\alpha\frac{d^2u}{d\,x^2}+\beta\,u+u^3=\psi,$$
$$u(0)=u(\pi)=u^{//}(0)=u^{//}(\pi)=0.$$

In B.M. Darinskii, Yu.I. Sapronov and S.L. Tsarev [3], studied bifurcation solutions and they found bif-spreadings of the following nonlinear system of algebraic equations,

$$\begin{aligned}
&\lambda_1\,\xi_1+\xi_1^3+2\xi_1\xi_2^2=0\\
&\lambda_2\,\xi_2+2\,\xi_1^2\,\xi_2+\xi_2^3=0.
\end{aligned}$$

And taking the example of the following equation with

$$\frac{d^4p}{d\,x^4}+\alpha\frac{d^2p}{d\,x^2}+\beta\,p+p^3=0$$
$$p(0)=p(\pi)=p^{//}(0)=p^{//}(\pi)=0.$$

$$\begin{aligned}
&\xi_1^3+2\,\xi_1\xi_2^2+5\,r\,\xi_1^2+4\,r\,\xi_2^2+\lambda_1\,\xi_1+q_1=0\\
&\xi_2^3+2\,\xi_1^2\,\xi_2+8\,r\,\xi_1\,\xi_2+\lambda_2\,\xi_2=0,
\end{aligned}$$

In M.J. Mohammed [11], studied bifurcation solutions and he found the bifurcation diagram of the following nonlinear system of algebraic equations,

And taking the example of the following equation with symmetry of $\psi(x)$

$$\frac{d^4w}{d\,x^4}+\alpha\frac{d^2w}{d\,x^2}+\beta\,w+w^2+w^3=\psi,$$
$$w(0)=w(\pi)=w^{//}(0)=w^{//}(\pi)=0.$$

In M.A. Abdul Hussain [9], studied bifurcation solutions and he found the bifurcation set of the following nonlinear system of algebraic equations,

$$\begin{aligned}
&\xi_1^3+2\xi_1\xi_2^2+5b\xi_1^2+4b\xi_2^2+\lambda_1\,\xi_1+\lambda_2\,\xi_2+q_1=0\\
&\xi_2^3+2\xi_1^2\,\xi_2+8b\,\xi_1\,\xi_2+\lambda_3\,\xi_1+\lambda_4\,\xi_2=0
\end{aligned}$$

And taking the example of the following equation with symmetry of $\psi(x)$

$$\frac{d^4w}{dx^4}+\alpha\frac{d^2w}{dx^2}+(\beta+\varepsilon_1 x)w+\varepsilon_2\frac{dw}{dx}+w^2+w^3=\psi,$$
$$w(0)=w(\pi)=w^{//}(0)=w^{//}(\pi)=0.$$

In A. A. Mizeal [1] , studied bifurcation solutions and he found the bifurcation diagram of the following nonlinear system of algebraic equations,

$$\xi_1^2+\xi_2^2+\lambda_1\xi_1+\lambda_2\xi_2+q_1=0$$
$$2\xi_1\xi_2+\lambda_3\xi_1+\lambda_4\xi_2=0$$

And taking the example of the following equation with symmetry of $\psi(x)$

$$\frac{d^4w}{dx^4}+\alpha\frac{d^2w}{dx^2}+(\beta+\varepsilon_1 x)w+\varepsilon_2\ w'+w^2=\psi,$$
$$w(0)=w(1)=w^{//}(0)=w^{//}(1)=0.$$

In M.A. Abdul Hussain [10], determined the bifurcation diagram of the specified problem of the following nonlinear system of algebraic equations,

$$A_1x_1^2+A_2x_2^2+A_3x_3^2+A_4x_1x_3+\lambda_1x_1-q_1=0$$
$$B_1x_1x_2+B_2x_2x_3+\lambda_2x_2=0$$
$$C_1x_1^2+C_2x_2^2+C_3x_3^2+C_4x_1x_3+\lambda_3x_3-q_3=0$$

And taking the example of the following equation

$$\delta\frac{d^4z}{dx^4}+\alpha\frac{d^2z}{dx^2}+\beta z+z^2=0,$$
$$z(0)=z(\pi)=z^{//}(0)=z^{//}(\pi)=0,$$

In A. K. Shanan, M.A. Abdul Hussain [2], studied bifurcation solutions and he determined the bifurcation diagram of the specified problem of the following nonlinear system of algebraic equations,

$$A_1x_1^2+A_2x_2^2+A_3x_3^2+A_4x_1x_3+\lambda_1x_1=0$$
$$B_1x_1x_2+B_2x_2x_3+\lambda_2x_2=0$$
$$C_1x_1^2+C_2x_2^2+C_3x_3^2+C_4x_1x_3+\lambda_3x_3=0$$

And taking the example of the following equation

$$\delta\frac{d^4z}{dx^4}+\alpha\frac{d^2z}{dx^2}+\beta z+z^2=0,$$
$$z(0)=z(\pi)=z^{//}(0)=z^{//}(\pi)=0,$$

In M.J. Mohammed [12], studied bifurcation solutions and he found the Discriminant set of the elastic beams equation for some values of,

$$\xi_1^2+\frac{16}{5}\xi_2^2+\lambda_1\xi_1+q_1=0$$
$$\frac{5}{4}\xi_1\xi_2+\lambda_2\xi_2=0$$

And taking the example of the following equation with symmetry of $\psi(x)$

$$\frac{d^4w}{dx^4}+\alpha\frac{d^2w}{dx^2}+\beta w+ww^{//}=\psi,$$
$$w(0)=w(\pi)=w^{//}(0)=w^{//}(\pi)=0.$$

In M.J. Mohammed [13], studied bifurcation solutions and he found the Discriminant set . Also, he calculated the topological indices of the solutions of the following nonlinear system of algebraic equations

$$\eta_1^2\eta_2 + b\eta_2^3 + \tilde{\lambda}_1\eta_1 - q_1 = 0$$
$$\eta_1^3 + b\eta_1\eta_2^2 + \tilde{\lambda}_2\eta_2 = 0$$

And taking the example of the following equation with symmetry of $\Psi(x)$

$$\mathbb{z}_{xxxx} + \alpha\mathbb{z}_{xx} + \beta\mathbb{z} + \mathbb{z}^2\mathbb{z}^{/} = \Psi,$$
$$\mathbb{z}(0) = \mathbb{z}(1) = \mathbb{z}^{//}(0) = \mathbb{z}^{//}(1) = 0$$

Our goal in this paper is finding the bifurcation diagram (Caustic) calculate the topological indices of the solutions of the following systems,

$$\xi_1^2 + \frac{16}{5}\xi_2^2 + \lambda_1\xi_1 + q_1 = 0$$
$$\frac{5}{4}\xi_1\xi_2 + \lambda_2\xi_2 + q_2 = 0 \qquad \text{…. (1.3)}$$

where,

$$\xi = (\xi_1, \xi_2), \quad \tilde{\lambda} = (\lambda_1, \lambda_2, q_1, q_2) \in R^4.$$

And,

$$\xi_1^3 + 2\xi_1\xi_2^2 + 5b\xi_1^2 + 4b\xi_2^2 + \lambda_1\xi_1 + \lambda_2\xi_2 + q_1 = 0$$
$$\xi_2^3 + 2\xi_1^2\xi_2 + 8b\xi_1\xi_2 + \lambda_3\xi_1 + \lambda_4\xi_2 + q_2 = 0 \qquad \text{…. (1.4)}$$

where,

$$\xi = (\xi_1, \xi_2), \quad \hat{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, q_1, q_2) \in R^6, \quad b = \frac{16}{45}\sqrt{\frac{2}{\pi}}.$$

2. Preliminaries

In this section, we will make some preliminaries to show our main results.

Definition 2.1 [3]

Let X and Y be real linear Banach spaces . Let $A: X \to Y$ be a linear continuous operator . Then A is a Fredholm operator if the space $Ker(A)$ and $Coker(A) = Y/Im(A)$ are finite dimensional . The number $ind(A) = dim\,Ker(A) - dim\,Coker(A)$ is called the (Fredholm) index of the operator A.

Definition 2.2[6]

Let $\mathcal{H}: R^m \to R^n$ be a C^1 map. A point $x_0 \in R^m$ is called a regular point of $\mathcal{H}$ if the Jacobian $D\mathcal{H}$ has maximal rank $\min\{p, q\}$. A value $y_0 \in R^n$ is called a regular value of $\mathcal{H}$ if x_0 is a regular point of $\mathcal{H}$ for all $x \in \mathcal{H}^{-1}(y_0)$. Points and values are called singular if they are not regular.

Definition 2.3[3]

The set of all λ in which equation (1.1) has degenerate solutions is called the Discriminant set(bifurcation set) of equation (1.1), denoted by $\sum$.

Definition 2.4 [3]

The Discriminate set together with the spreading of the regular solutions in the space of parameters is called the Bifurcation diagram (Caustic).

Definition 2.5 [5]

Nonlinear algebraic equation which are also called polynomial equations, are defined by equating polynomials to zero.

Definition 2.6 [5]

The general form of a system of Nonlinear algebraic equations can be defined by the following formula

$$f(x) = 0$$

where $f(x)$ is a vector function of x i.e there are n equations can be written in expanded form as

$f_1(x_1, x_2, x_3, \ \dots x_n) = 0$, $f_2(x_1, x_2, x_3, \ \dots x_n) = 0$, …., $f_n(x_1, x_2, x_3, \ \dots x_n) = 0$

Where f is a vector function of Variable x.

Theorem 2.1 [3]

The point $s \in X$ is a solution of the equation $L(x, \lambda) = 0$ if and only if, $s = \sum_{i=1}^{n} \overline{x}_i e_i + \Theta(\overline{\eta}, \overline{\lambda})$, where, $\overline{\eta}$ is a solution of equation $\phi(\eta, \widetilde{\lambda}) = 0$, $\eta = (x_1, x_2, ..., x_n)$.

3. Bifurcation Diagrams

3.1 Bifurcation analysis of nonlinear system with four parameters

In [12] the author found the Discriminant set (bifurcation set) of the elastic beams equation for some values of parameters of nonlinear system (1.3) with symmetry that is; the function ψ is a symmetric with respect to the involution $I : \psi(x) \mapsto \psi(\pi - x)$. In this section we study the bifurcation analysis of nonlinear system of algebraic equations (1.3) if the function ψ is not symmetric, then we have $q_2 \neq 0$, this means that the space of parameters consisted of another parameter different from the parameters λ_1, λ_2 and q_1, which play a role in the solutions of nonlinear system (1.3). Thus, from system (1.3), we have the following parameterization,

$$\begin{aligned} q_1 &= -(\xi_1^2 + \frac{16}{5}\xi_2^2 + \lambda_1 \xi_1) \\ q_2 &= -(\frac{5}{4}\xi_1 \xi_2 + \lambda_2 \xi_2). \end{aligned} \qquad \text{…. (3.1)}$$

To study the Discriminant set of the above nonlinear system it is convenient to fix the value of λ_2 and the Bifurcation diagram (Caustic) in the space of parameters (λ_1, q_1, q_2) can be described by the following graph, (all figures were drawn by Maple 13).

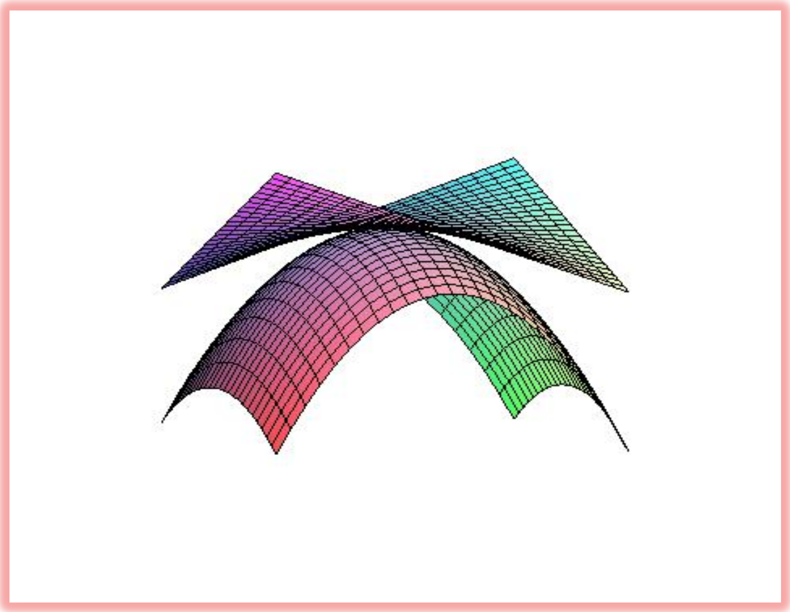

Fig. 1 Describes Caustic in the space of parameters (λ_1, q_1, q_2).

From Figure (1), we can obtain many sections in the plane of parameters, but we are interested only in the Caustic. So by fixing the values of λ_1 and λ_2 we have the following sections of Caustic as λ_1 and λ_2 changes,

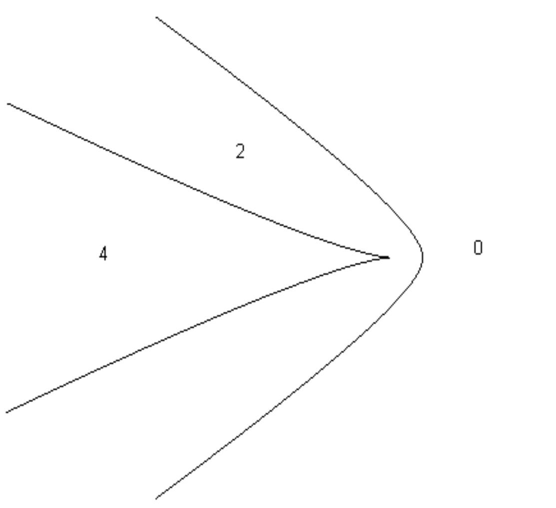

Fig. 2: Described Caustic in the $q_1 q_2$- plane when $\lambda_1 = 0$, $\lambda_2 \in (-\infty, \infty) \setminus [-0.7, 0.7]$, or $\lambda_2 = 0$, $\lambda_1 \in (-\infty, \infty) \setminus [-1.2, 1.2]$.

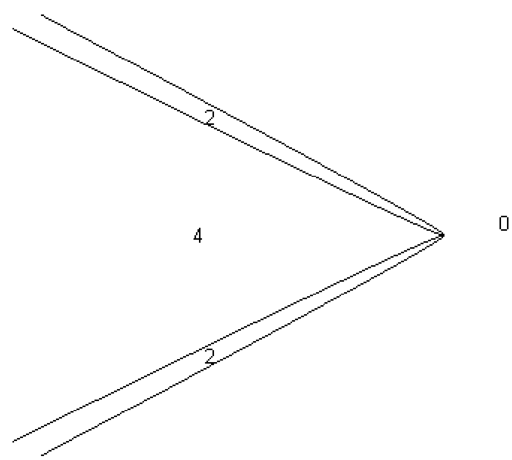

Fig. 3: Described Caustic in the $q_1 q_2$- plane when $\lambda_1 = 0$, $\lambda_2 \in [-0.7, 0.7] \setminus \{0\}$, or $\lambda_2 = 0$, $\lambda_1 \in [-1.2, 1.2] \setminus \{0\}$.

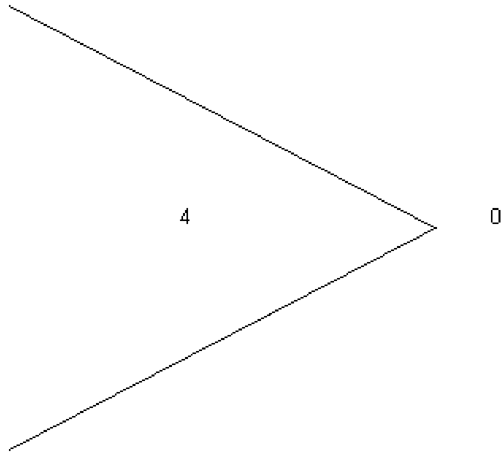

Fig. 4: Described Caustic in the $q_1\, q_2$- plane when $\lambda_1 = 0,\ \lambda_2 = 0.$

In figures (3) and (4) the complement of the Discriminant set $\Omega = R^4 \setminus \Sigma$ is the union of three open subsets $\Omega = \Omega_0 \cup \Omega_2 \cup \Omega_4$ such that if $\tilde{\lambda} \in \Omega_0$ then the nonlinear system(3.1) has no regular solutions, if $\tilde{\lambda} \in \Omega_2$ then the nonlinear system(3.1) has two regular solutions with topological indices 1, -1 and if $\tilde{\lambda} \in \Omega_4$ then nonlinear system(3.1) has four regular solutions with topological indices 1,-1,1,-1. In figures (4), the complement of the Discriminant set is the union of two connected open subsets $\Omega_0 \cup \Omega_4$, every region has a fixed number of regular solutions such that if $\tilde{\lambda} \in \Omega_0$ then the nonlinear system(3.1) has no regular solutions, if $\tilde{\lambda} \in \Omega_4$ then the nonlinear system(3.1) has four regular solutions with topological indices1,-1,1,-1.

3.2 Bifurcation analysis of nonlinear system with six parameters

The bifurcation solutions of nonlinear system (1.4) when $q_2 \neq 0$ have been studied by Abdul Hussain [9], in his work he found the Discriminant set (bifurcation set) of the nonlinear system (1.4) with symmetry of the function ψ. In this section we studied the bifurcation analysis of nonlinear system of algebraic equations (1.4) if the function ψ is not symmetric, then we have $q_2 \neq 0$, this means that the space of parameters consisted of another parameter different from the parameters $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and q_1, which plays a role in the solutions of nonlinear system (1.4). Thus, from system (1.4), we have the following parameterization,

$$q_1 = -(\xi_1^3 + 2\xi_1\xi_2^2 + 5b\,\xi_1^2 + 4b\,\xi_2^2 + \lambda_1\,\xi_1 + \lambda_2\,\xi_2)$$
$$q_2 = -(\xi_2^3 + 2\xi_1^2\,\xi_2 + 8b\,\xi_1\,\xi_2 + \lambda_3\,\xi_1 + \lambda_4\,\xi_2) \qquad \text{…. (3.2)}$$

To study the Discriminant set of the above nonlinear system, it is convenient to fix the values of $\lambda_2, \lambda_3, \lambda_4$ and the Geometric description of the caustic in the space of parameters (λ_1, q_1, q_2) is given by the following,

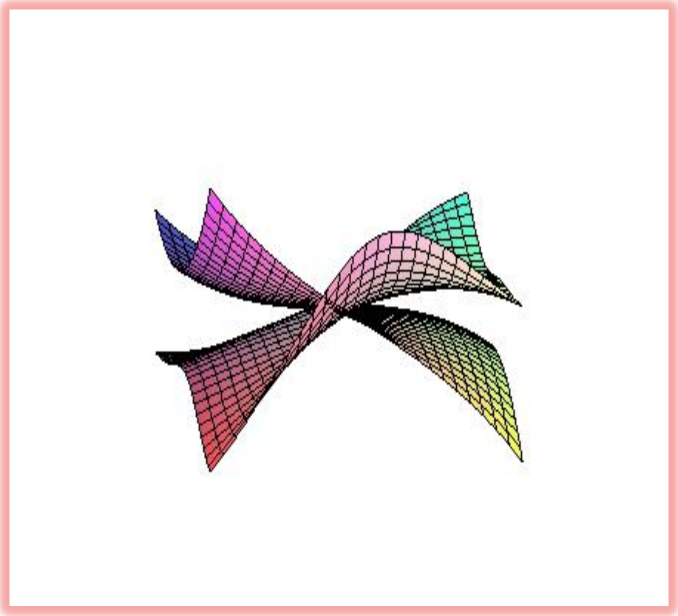

Fig. 5 Describes Caustic in the space of parameters (λ_1, q_1, q_2).

Theoretically, it is not easy to find the solutions of nonlinear system (3.2), so we use program (Maple 13 and Mathematica 6.0) to find all the solutions of nonlinear system (3.2). The maximum number of the solutions of the nonlinear system equation (3.2) is nine. From Figure (5), we can obtain many sections in the plane of parameters, but we are interested only in the Caustic. So by fixing the values of $\lambda_1, \lambda_2, \lambda_3$ and λ_4 we have the following sections of Caustic as $\lambda_1, \lambda_2, \lambda_3$ and λ_4 changes,

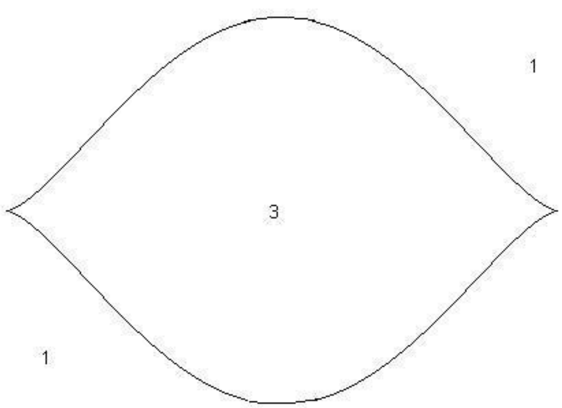

Fig. 6: Described Caustic in the $q_1 q_2$- plane when $\lambda_1 = 60,\ \lambda_2 = 0,\ \lambda_3 = 0,\ \lambda_4 = 0.$

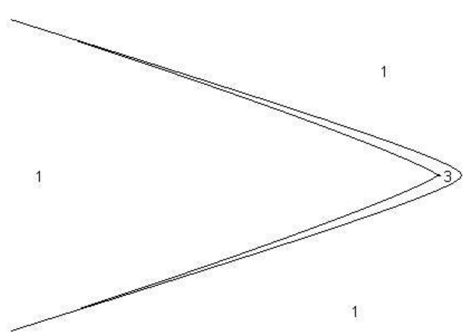

Fig. 7: Described Caustic in the $q_1 q_2$- plane when $\lambda_1 = 0.64,\ \lambda_2 = 0,\ \lambda_3 = 0,\ \lambda_4 = 0.64.$

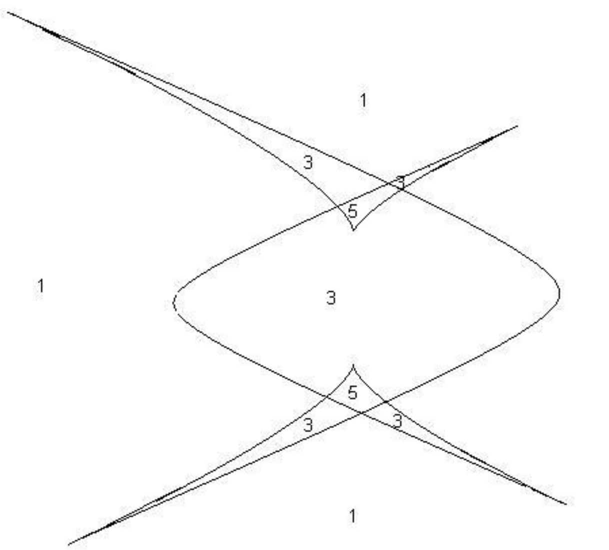

Fig. 8: Described Caustic in the $q_1 q_2$- plane when $\lambda_1 = -12,\ \lambda_2 = 0,\ \lambda_3 = -0.6,\ \lambda_4 = 1.$

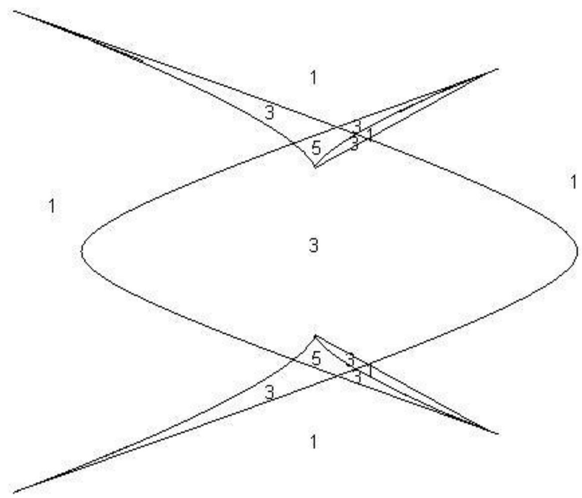

Fig. 9: Described Caustic in the $q_1 q_2$- plane when $\lambda_1 = -17,\ \lambda_2 = 0,\ \lambda_3 = 0,\ \lambda_4 = 3.84.$

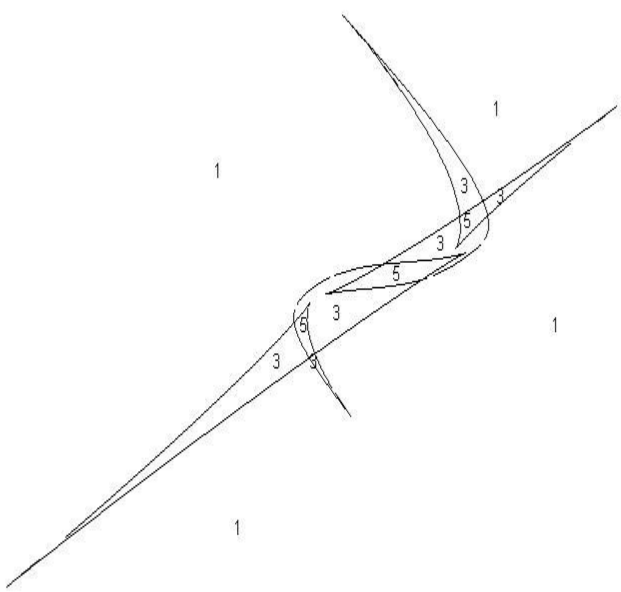

Fig. 10: Described Caustic in the $q_1 q_2$- plane when $\lambda_1 = -4.3,\ \lambda_2 = 2.5,\ \lambda_3 = -1.5,\ \lambda_4 = -0.3.$

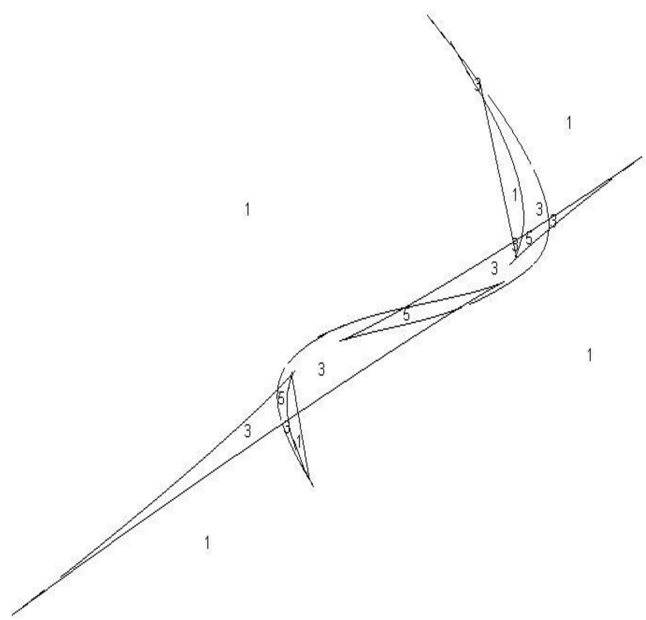

Fig. 11: Described Caustic in the $q_1 q_2$- plane when $\lambda_1 = -5.3,\ \lambda_2 = 3.5,\ \lambda_3 = -2.5,\ \lambda_4 = 0.7.$

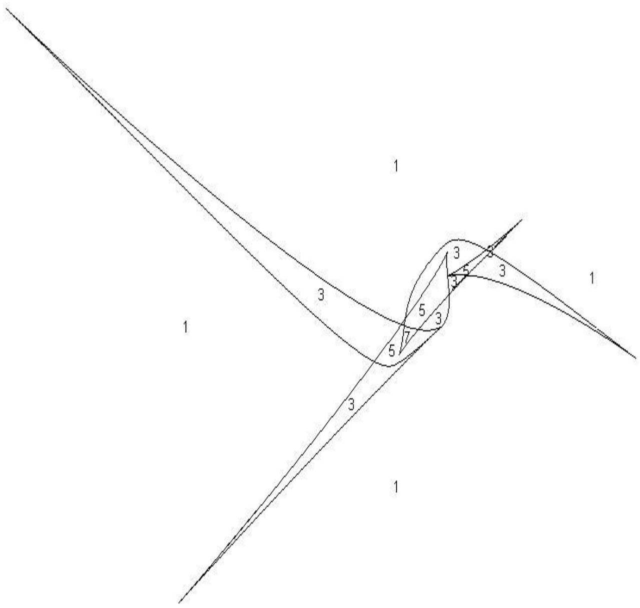

Fig. 12: Described Caustic in the $q_1 q_2$- plane when $\lambda_1 = 0.2505,\ \lambda_2 = -0.3505,\ \lambda_3 = 0.2505,\ \lambda_4 = -0.3505.$

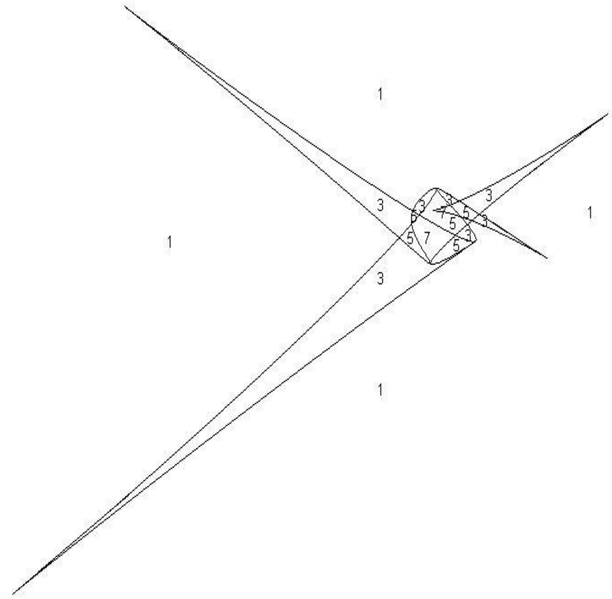

Fig. 13: Described Caustic in the $q_1 q_2$- plane when $\lambda_1 = 0.088,\ \lambda_2 = -0.028,\ \lambda_3 = 0.172,\ \lambda_4 = -0.112.$

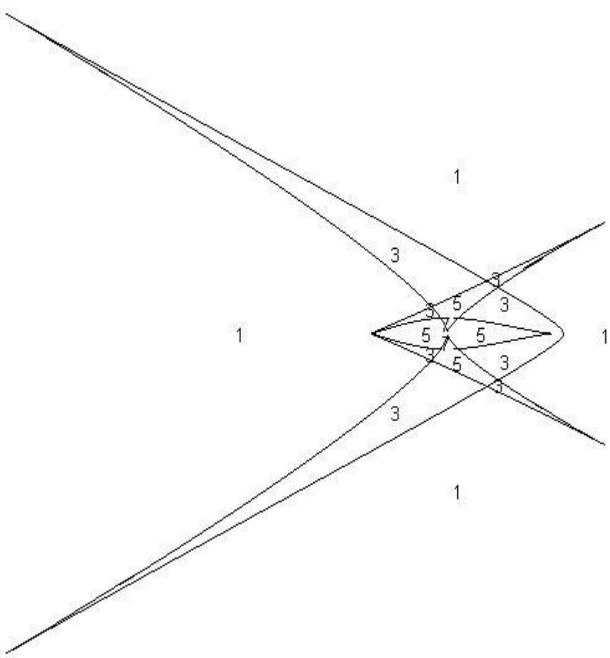

Fig. 14: Described Caustic in the $q_1 q_2$- plane when $\lambda_1 = -0.7,\ \lambda_2 = 0,\ \lambda_3 = 0,\ \lambda_4 = 0.$

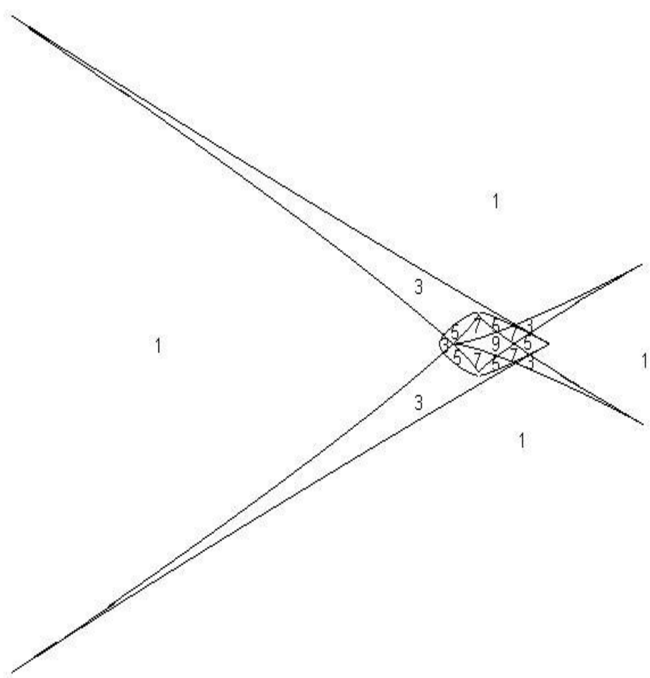

Fig. 15: Described Caustic in the $q_1 q_2$- plane when $\lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = 0,\ \lambda_4 = 0.$

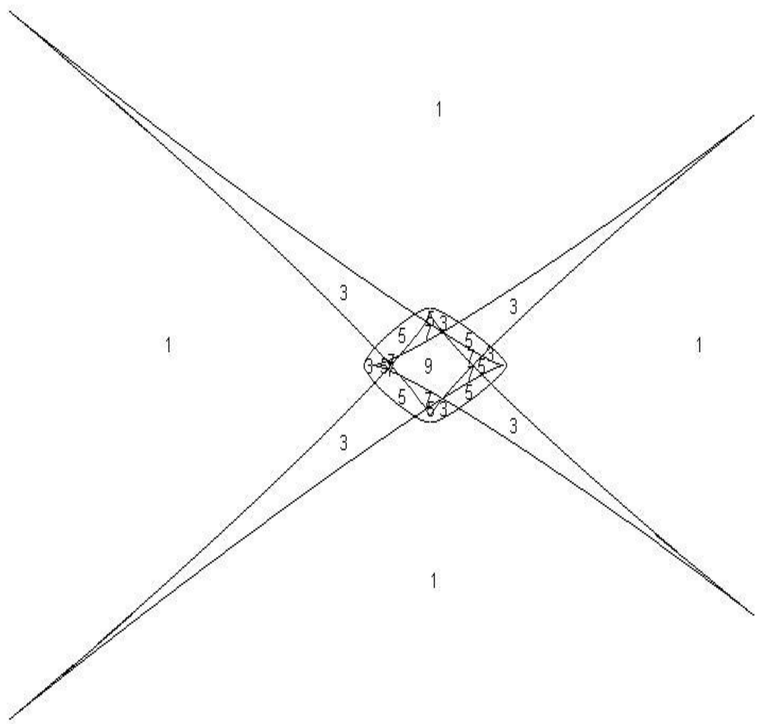

Fig. 16: Described Caustic in the $q_1\, q_2$- plane when $\lambda_1 = -10,\ \lambda_2 = 0,\ \lambda_3 = 0,\ \lambda_4 = -10.$

In figures (6) and (7) the complement of the Discriminant set $\Omega = R^6 \backslash \Sigma$ is the union of two open subsets $\Omega = \Omega_1 \cup \Omega_3$, every region has a fixed number of regular solutions such that if $\hat{\lambda} \in \Omega_1$ then the nonlinear system(3.2) has one regular solution with topological indices -1 and if $\hat{\lambda} \in \Omega_3$ then equation (3.2) has three regular solutions with topological indices -1,1,-1. In figures (8), (9), (10) and (11). The complement of the Discriminant set is the union of three open subsets $\Omega = \Omega_1 \cup \Omega_3 \cup \Omega_5$ such that if $\hat{\lambda} \in \Omega_1$ then the nonlinear system(3.2) has one regular solution with topological indices -1, if $\hat{\lambda} \in \Omega_3$ then system (3.2) has three regular solutions with topological indices -1,1,-1 and if $\hat{\lambda} \in \Omega_5$ then system (3.2) has five regular solutions with topological indices -1,1,-1,1,-1. In figures (12), (13) and (14). The complement of the Discriminant set is the union of four open subsets $\Omega = \Omega_1 \cup \Omega_3 \cup \Omega_5 \cup \Omega_7$ such that if $\hat{\lambda} \in \Omega_1$ then the nonlinear system(3.2) has one regular solution with topological indices -1, if $\hat{\lambda} \in \Omega_3$ then system (3.2) has three regular solutions with topological indices -1,1,-1, if $\hat{\lambda} \in \Omega_5$ then system (3.2) has five regular solutions with topological indices -1,1,-1,1,-1 and if $\hat{\lambda} \in \Omega_7$ then system (3.2) has seven regular solutions with topological indices -1,1,-1,1,-1,1,-1. In figures (15) and (16). The complement of the Discriminant set is the union of five open subsets $\Omega = \Omega_1 \cup \Omega_3 \cup \Omega_5 \cup \Omega_7 \cup \Omega_9$ such that if $\hat{\lambda} \in \Omega_1$ then the nonlinear system(3.2) has one regular solution with topological indices -1, if $\hat{\lambda} \in \Omega_3$ then system (3.2) has three regular solutions with

topological indices -1,1,-1, if $\hat{\lambda} \in \Omega_5$ then system (3.2) has five regular solutions with topological indices -1,1,-1,1,-1, if $\hat{\lambda} \in \Omega_7$ then system (3.2) has seven regular solutions with topological indices -1,1,-1,1,-1,1,-1 and if $\hat{\lambda} \in \Omega_9$ then system (3.2) has nine regular solutions with topological indices -1,1,-1,1,-1,1,-1,1,-1.

4. Applications

In this section, we give two examples about the nonlinear beams equation of the fourth order which describe the oscillations and motion of wave on elastic foundations. we apply the general method of Lyapunov-Schmidt to study the bifurcation analysis of the fourth order nonlinear ODE,

Example 4.1 Consider the nonlinear beams equation,

$$\frac{d^4 w}{d x^4} + \alpha \frac{d^2 w}{d x^2} + \beta w + w w^{//} = \psi, \quad \text{.... (4.1)}$$

with the boundary conditions,

$$w(0) = w(\pi) = w''(0) = w''(\pi) = 0.$$

Where w is the deflection of beam, $w'' = \frac{d^2 w}{dx^2}$ and $\psi = \varepsilon \varphi(x)$ (ε - small parameter) is a continuous function is not symmetric function with respect to the involution that is; $q_2 \neq 0$..

Equation (4.1) can be rewritten in the form of operator equation, that is,

$$\mathrm{L}(w,\lambda) = \frac{d^4 w}{d x^4} + \alpha \frac{d^2 w}{d x^2} + \beta w + w w^{//} \quad \text{...(4.2)}$$

where L*: X*→*Y* is a nonlinear Fredholm map of index zero from Banach space X to Banach space Y, $X = \{w(x) \in C^4([0,\pi], R): w(0) = w(\pi) = w^{//}(0) = \mathbb{z}^{//}(\pi) = w^4(0) = w\mathbb{z}^4(\pi) = 0\}$, $Y = \{w(x) \in C^0([0,\pi], R)\}$ and w= $w(x)$, $\lambda = (\alpha, \beta)$. In this case, the bifurcation solutions of equation (4.1) is equivalent to the bifurcation solutions of the operator equation,

$$\mathrm{L}(w,\lambda) = \psi, \quad \psi \in Y. \quad \text{....(4.3)}$$

Theorem 4.1 The bifurcation equation corresponding to problem (4.1) without Symmetry of $\psi(x)$ is given by the following nonlinear system of two quadratic equations,

$$\Phi(\xi, \tilde{\lambda}) = \begin{pmatrix} \xi_1^2 + \frac{16}{5}\xi_2^2 + \lambda_1 \xi_1 + q_1 \\ \frac{5}{4}\xi_1 \xi_2 + \lambda_2 \xi_2 + q_2 \end{pmatrix} + o(|\xi|^2) + O(|\xi|^2) O(\delta) = 0\,.$$

Where,

$$\xi = (\xi_1, \xi_2), \quad \tilde{\lambda} = (\lambda_1, \lambda_2, q_1, q_2) \in R^4, \quad \delta = (\delta_1, \delta_2).$$

Furthermore, the solutions of the equation in (4.1) are in one-to-one corresponding with the solutions of the nonlinear system.

The proof of Theorem (4.1) is similar to the proof of Theorem (2.1) in [12]; therefore , we omit it.

he Discriminant set (The bifurcation set) Σ of the map $\Phi(\xi, \tilde{\lambda})$ is locally equivalent in the

neighbourhood of point zero to the Discriminant set of the map $\Phi_1(\xi,\tilde{\lambda})$ [15],

$$\Phi_1(\xi,\tilde{\lambda}) = \begin{pmatrix} \xi_1^2 + \frac{16}{5}\xi_2^2 + \lambda_1\xi_1 + q_1 \\ \frac{5}{4}\xi_1\xi_2 + \lambda_2\xi_2 + q_2 \end{pmatrix}. \quad , \qquad \dots (4.4)$$

this means that, to study the Discriminant set of the map $\Phi(\xi,\tilde{\lambda})$ it is sufficient to study the Discriminant set of the map $\Phi_1(\xi,\tilde{\lambda})$. The point $a \in E$ is a solution of equation (4.3) if and only if

$$a = \sum_{i=1}^{2} \bar{\xi}_i e_i + \Theta(\bar{\xi},\bar{\lambda}),$$

where $\bar{\xi}$ is a solution of equation

$$\Phi_1(\xi,\tilde{\lambda}) = 0. \qquad \dots (4.5)$$

This mean that, to study the Discriminant set of the map $\Phi(\xi,\tilde{\lambda})$ it is sufficient to study the Discriminant set of the map $\Phi_1(\xi,\tilde{\lambda})$.

Example 4.2

The oscillations and motion of wave of the elastic beams on elastic foundations can be described by means of the following ODE,

$$\frac{d^4w}{dx^4} + \alpha\frac{d^2w}{dx^2} + (\beta+\varepsilon_1 x)w + \varepsilon_2\frac{dw}{dx} + w^2 + w^3 = \psi, \qquad \dots (4.6)$$

with the boundary conditions,

$$w(0) = w(\pi) = w''(0) = w''(\pi) = 0.$$

where ε_1 and ε_2 are small parameters indicate the perturbation and $\psi = \varepsilon\varphi(x)$ (ε – small parameter) is not symmetric function with respect to the involution $I : \psi(x) \mapsto \psi(\pi - x)$ implies that $q_2 \neq 0$.

Suppose that $L : X \to Y$ is a nonlinear Fredholm operator of index zero from Banach space X to Banach space Y, where $X = C^4([0,\pi],R)$ is the space of all continuous functions that have derivative of order at most four, $Y = C^0([0,\pi],R)$ is the space of all continuous functions and L is given in the form of operator equation:

$$L(w,\lambda,\varepsilon_1,\varepsilon_2) = \frac{d^4w}{dx^4} + \alpha\frac{d^2w}{dx^2} + (\beta+\varepsilon_1 x)w + \varepsilon_2\frac{dw}{dx} + w^2 + w^3 \qquad \dots (4.7)$$

where $w=w(x)$, $x \in [0,\pi]$, $\lambda = (\alpha,\beta)$. Every solution of equation (4.7) is a solution of operator equation,

$$L(w,\lambda,\varepsilon_1,\varepsilon_2) = \psi, \quad \psi \in Y. \qquad \dots (4.8)$$

Theorem 4.2 The bifurcation equation $\Phi(\xi,\lambda,\varepsilon_1,\varepsilon_2) = f^{(2)}(u+\Theta(u,\lambda),\lambda,\varepsilon_1,\varepsilon_2) = \psi_1$ corresponding to the equation (4.8) without Symmetry of $\psi(x)$ have the following form,

$$\Phi(\xi,\hat{\lambda}) = \begin{pmatrix} \xi_1^3 + 2\xi_1\xi_2^2 + 5b\xi_1^2 + 4b\xi_2^2 + \lambda_1\xi_1 + \lambda_2\xi_2 + q_1 \\ \xi_2^3 + 2\xi_1^2\xi_2 + 8b\xi_1\xi_2 + \lambda_3\xi_1 + \lambda_4\xi_2 + q_2 \end{pmatrix} + o(|\xi|^3) + O(|\xi|^3)O(\delta) = 0.$$

where,

$$\xi = (\xi_1,\xi_2), \quad \hat{\lambda} = (\lambda_1,\lambda_2,\lambda_3,\lambda_4,q_1,q_2) \in R^6,$$

$$\delta = (\delta_1,\delta_2), \quad b = \frac{16}{45}\sqrt{\frac{2}{\pi}}.$$

The proof of Theorem (4.2) is similar to the proof of Theorem (2.1) in [8]; therefore , we omit it.

The discriminant set Σ of the map $\Phi(\xi,\hat{\lambda})$ is locally equivalent in the neighbourhood of point zero to the discriminant set of the map $\Phi_1(\xi,\hat{\lambda})$,

$$\Phi_1(\xi,\hat{\lambda}) = \begin{pmatrix} \xi_1^3 + 2\xi_1\xi_2^2 + 5b\xi_1^2 + 4b\xi_2^2 + \lambda_1\xi_1 + \lambda_2\xi_2 + q_1 \\ \xi_2^3 + 2\xi_1^2\xi_2 + 8b\xi_1\xi_2 + \lambda_3\xi_1 + \lambda_4\xi_2 + q_2 \end{pmatrix} \quad \dots (4.9)$$

this means that, to study the discriminant set of the map $\Phi(\xi,\hat{\lambda})$ it is sufficient to study the discriminant set of the map $\Phi_1(\xi,\hat{\lambda})$. The point $a \in E$ is a solution of equation (4.8) if and only if

$$a = \sum_{i=1}^{2} \bar{\xi}_i e_i + \Theta(\bar{\xi},\bar{\lambda}),$$

where, $\bar{\xi}$ is a solution of equation

$$\Phi_1(\xi,\hat{\lambda}) = 0. \quad \dots (4.10)$$

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مخطط التفرع لنظام غير خطي من المعادلات الجبرية مع معلمات

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الخلاصة

درسنا في هذا البحث حلول التفرع لمعادلة التفرع (نظام غير خطي من المعادلات الجبرية مع أربعة أو ستة معلمات). أيضا ، وجدنا وصفاً هندسي جديد للمجموعة المميزة مع انتشار الحلول المنتظمة لكل منطقة.