COMPLETELY SEMI ESSENTIAL FUZZY SUBMODULE

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Abstract

In this paper, we introduce the notion of a completely semi essential fuzzy submodule which connect between the concept of essential fuzzy submodule and semi essential fuzzy submodule and we give some properties of this concept.

Key words essential (fuzzy) submodule, semi essential (fuzzy) submodule, completely semi essential (fuzzy) submodule .

1 Introduction

The concept of fuzzy sets was introduced by Zadeh in 1965 [9]. It was first applied in the theory of groups by Rosenfeld in 1971 [8] . The concept of fuzzy module was introduced by Negoita and Relescu in 1975 [4]. Since then several authors have studied fuzzy modules. The concept of essential fuzzy submodule was introduced by Hadi 2000[2] . In this paper Let R be a commutative ring with unity and let M be unitary R-module . we list some known properties of essential submodules and we present some known characterizations of essential submodules which are relative to our work. we introduce a completely semi essential submodule concept. Also we generalize some known properties of essential fuzzy submodules to completely semi essential fuzzy submodules.

2 The Basic Concepts

In this section ,we sall review some concepts which we shall used later .

Definition (2.1) [9]Let S be a non-empty set and I be the closed interval [0, 1] of the real line (real numbers). *A fuzzy set* A in S (a fuzzy subset of S) is a function from S into I .

Definition (2.2) [8] Let $x_t : S \to [0, 1]$ be a fuzzy subset of S, where $x \in S$, $t \in [0, 1]$ defined by : $x_t(y) = \begin{cases} t & if\ y = x \\ 0 & if\ y \neq x \end{cases}$, for all $y \in S$.

x_t is called *a fuzzy singleton* or a fuzzy point in S .

If x=0 and t=1 then:

$0_1(y) = \begin{cases} 1 & if\ y = 0 \\ 0 & if\ y \neq 0 \end{cases}$, for all y∈ S

is called the *fuzzy zero singleton*.

Definition (2.3) [10] Let A and B be two fuzzy subsets of S .Then

(1) A= B if and only if A(x) = B (x), for all x ∈ S.

(2) A ⊆ B if and only if A (x) ≤ B (x) , for all x ∈ S .

If A ⊆ B and there exists x ∈S such that A (x) < B (x), then A is called ***a proper fuzzy*** subset of B and written A ⊂ B .

By part (2) we can deduce that $x_t \subseteq A$ if and only if A(x)≥t .

Definition (2.4) [10] Let A and B be two fuzzy subsets of S .Then

(1) (A ∩ B) (x) = min {A (x), B (x)}, for all x ∈ S.
(2) (A ∪ B) (x) = max {A (x), B (x)}, for all x ∈ S.

Where A ∩ B and A ∪ B are fuzzy subsets of S .

Definition (2.5) [6], [7] Let A be a fuzzy subset of S, for all t ∈ [0, 1] , the set A_t = { x ∈ S , A(x) ≥ t} is called *a level subset* of A.

Remark (2.6) [6], [7] The following properties of a level subsets hold for each t ∈ [0, 1] :

(1) $(A \cap B)_t = A_t \cap B_t$,
(2) $(A \cup B)_t = A_t \cup B_t$ and
(3) A = B if and only if $A_t = B_t$.

Definition (2.7) [6], [7] Let M be an R-module. A fuzzy set A of M is called *a fuzzy module* of an R-module M if :-

(1) A (x−y) ≥ min { A (x), A (y)}, for all x, y ∈ M.

(2) A (rx)≥A (x), for all x∈ M and r ∈ R.

(3) A (0) = 1(0 is the zero element of M).

Definition (2.8) [6],[11] Let A and B be two fuzzy modules of an R-module M . B is called *a fuzzy submodule* of A if B ⊆ A .

Proposition (2.9) [7] Let N be a fuzzy subset of an R-module M. Then the level subset N_t , t ∈ [0, 1] is a submodule of M if and only if N is a fuzzy submodule of A where A is a fuzzy module of an R-module M .

Definition (2.10) [7] If N is a fuzzy module of an R-module M , then the submodule N_t of M is called *the level submodule* of M where t ∈ [0, 1] .

Proposition (2.11) [7] Let A be a fuzzy module of an R-module M. Let {N_α : α ∈ Λ} be a family of fuzzy submodules of A .Then

(1) $\left(\bigcap_{\alpha\in\Lambda} N_\alpha\right)$ is a fuzzy submodule of A.
(2) If {N_α : α ∈ Λ} is a chain ,Then $\left(\bigcup_{\alpha\in\Lambda} A_\alpha\right)$ is a fuzzy submodule of A .

Definition (2.12) [5] Let A be a fuzzy module . A proper fuzzy submodule N of A is called *a prime fuzzy submodule* whenever $r_t x_k \subseteq N$ for fuzzy singleton r_t of

R and $x_k \subseteq A$ implies to either $r_t \in (N : A)$ or $x_k \subseteq N$, where :

$(N : A) = \{r_t : r_t A \subseteq N, r_t$ fuzzy singleton of R$\}$.

Recall that a submodule P in the sense ordinary which is called *prime submodule* of an R-module M if $rx \in P$ for $r \in R$ and $x \in M$ then either $x \in P$ or $r \in (P:M)$ where $(P:M)=\{ r \in R : rM \subseteq P \}$,[3] .

Proposition (2.13) [5] Let A be a fuzzy module of an R-module M and let A(x) =1 for all $x \in M$ and N be a fuzzy submodule of A ,

Define $N: M \longrightarrow [0,1]$, by :

$N(x) = \begin{cases} 1 \text{ if } x \in P \\ k \text{ if } x \notin P \end{cases}$ such that $k \in (0,1]$.

Then N is a prime fuzzy submodule of A if and only if P is a prime submodule of M .

Definition (2.14) [2] Let A be a fuzzy module of an R-module M and let N be a fuzzy submodule of A .Then N is called an essential fuzzy submodule of A if $N \cap U = 0_1$ implies to $U=0_1$, for all fuzzy submodule U of A .

Definition (2.15) [2] Let A be a fuzzy module of an R-module M. Then a fuzzy submodule N of A is said to be *a Semi prime fuzzy submodule* if $r_t^2 x_s \subseteq N$ for all fuzzy singleton r_t of R , $x_s \subseteq N$ then $r_t x_s \subseteq N$.

Remark (2.16) [2] Every prime fuzzy submodule N of a fuzzy module A is a semi prime fuzzy submodule of A .

Definition (2.17) [3] ***:*** Let A be a fuzzy module of an R-module M and N be a non-zero fuzzy submodule of A .Then N is called *a semi essential fuzzy submodule* of A if for each fuzzy prime submodule P of A such that $N \cap P = 0_1$, then $P=0_1$.

Remark (2.18) [3] Each essential fuzzy submodule is a semi essential fuzzy submodule .

Definition(2.19) [1] : A non-zero submodule B of an R-module M is called *a 2-semi essential submodule* of M if $B \cap S = \{0\}$ then $S=\{0\}$ for each a semi prime submodule S of M .

Proposition (2.20) [1] Every 2- semi essential submodule of an R-module M is a semi essential submodule M .

Definition (2.21) [2] Let A be a fuzzy module of an R-module M .Then a fuzzy submodule N of A is said to be a completely prime if $r_t x_s \subseteq N$, $r_t \neq 0_t$ implies that $x_s \subseteq N$,,for all fuzzy singleton r_t of R , $x_s \subseteq A$.

Proposition (2.22)[2] Every completely prime fuzzy submodule of a fuzzy module A is a prime fuzzy submodule .

3 The Maim Results

In this section we shall introduce the definition of completely semi essential

fuzzy submodule and shall give some properties .

Definition (3.1) Let N be a fuzzy submodule of a fuzzy module A of an R-module M , N is called a completely semi essential fuzzy submodule of A if $N \cap C = 0_1$ then $C = 0_1$ for all completely prime fuzzy submodule C of A .

Example (3.2) Let A be a fuzzy module of a Z-module Z^2 with scalar multiplication (a,b)r=(ar,br) defined as :

$$A((a,b)) = \begin{cases} 1 & \text{if } a+b \in Ze \\ 1/2 & \text{if } a+b \in Zo \end{cases} ,$$

$$N((a,b)) = \begin{cases} 1 & \text{if } a,b \in Ze \\ 1/3 & \text{if } a,b \in Zo \\ 1/4 & \text{otherwise} \end{cases}$$

N is a completely semi essential fuzzy submodule of A.

Proposition (3.3) Let A be a fuzzy module of an R-module M and let N_1 and N_2 be two fuzzy submodules of A such that N_1 is a fuzzy submodule of N_2 , if N_1 is a completely semi essential fuzzy submodule of A then N_2 is a completely semi essential fuzzy submodule of A .

Proof :

Let C be a completely prime fuzzy submodule of A such that $N_2 \cap C = 0_1$

$N_1 \cap C = 0_1$ (Since $N_1 \subseteq N_2$) .

$C = 0_1$ (Since N_1 is a completely semi essential fuzzy submodule of A) .

N_2 is a completely semi essential fuzzy submodule of A

Corollary (3.4) Let A be a fuzzy module of an R-module M and let N_1 and N_2 be two fuzzy submodules of A such that N_1 is a fuzzy submodule of N_2 if N_1 is an essential fuzzy submodule of A then N_2 is a completely semi essential fuzzy submodule of A .

Proof :

Let C be a completely prime fuzzy submodule of A such that $N_2 \cap C = 0_1$

$N_1 \cap C = 0_1$ (Since $N_1 \subseteq N_2$) .

$C = 0_1$ (Since N_1 is an essential fuzzy submodule of A) .

Thus, N_2 is a completely semi essential fuzzy submodule of A .

Proposition (3.5) Let A be a fuzzy module of an R-module M and let N_1 and N_2 be two fuzzy submodule of A , if $N_1 \cap N_2$ is a completely semi essential fuzzy submodule of A then N_1 and N_2 are completely semi essential fuzzy submodules of A .

Proof:

Let $N_1 \cap N_2$ be a completely semi essential fuzzy submodule of A.

$N_1 \cap N_2 \subseteq N_1$ and $N_1 \cap N_2 \subseteq N_2$

Then N_1 and N_2 are completely semi essential fuzzy submodules of A (By proposition (3.3)) .

Proposition (3.6) Let A be a fuzzy module of an R-module M and N_i are completely semi essential fuzzy submodules of A for all i=1,2,…,n, if there exist one of these submodules is an essential fuzzy then $\bigcap_{i=1}^{n} N_i$ is a completely semi essential fuzzy submodule of A .

Proof:

If n=1 then it is true from hypothesis. Suppose it true if n=k-1 this implies to $\bigcap_{i=1}^{k-1} N_i$ is a completely semi essential fuzzy submodule of A. Let n=k and let be C is a completely prime fuzzy submodule of A, such that $\bigcap_{i=1}^{k} N_i \cap C=0_1$ $N_{i=k0}\cap[\bigcap_{i=1}^{k-1} N_i \cap C]=0_1$,where $N_{i=k0}$ is an essential fuzzy submodule of A. $\bigcap_{i=1}^{k-1} N_i \cap C=0_1$ (Since $N_{i=k0}$ is an essential fuzzy submodule of A). $C=0_1$ ($\bigcap_{i=1}^{k-1} N_i$ is a completely semi essential fuzzy submodule of A). Thus, the proof is completed .

Proposition (3.7) Let N be a fuzzy submodule of a fuzzy module A, if N is a 2-semi essential fuzzy submodule of A .Then N_t is a completely semi essential submodule of M ,where $t \in (0,1]$.

Proof :

Let P be a completely prime submodule of M and $N_t \cap P=\{0\}$.

P be a prime submodule of M (By proposition (2.22)).

Set $U(x)=\begin{cases} 1 \text{ if } x \in P \\ k \text{ if } x \notin P \end{cases}$, For all $k\in(0,1]$, $x\in M$ and $k\leq t$

Let us take $t\in(0,1]$, It is clear that $U_t=P$.

U is a prime fuzzy submodule of A (By proposition (2.13)).

Now , $N_t\cap P= N_t\cap U_t=\{0\}$

$(N\cap U)_t=(0_1)_t$ (By remark (2.6.1))

$N\cap U=0_1$ (By remark (2.6.3))

Then U is a semi prime fuzzy submodule (By proposition (2.16)).

$U=0_1$ (since N is a 2-semi essential fuzzy submodule of A).

Thus $P=U_t=(0_1)=\{0\}$

Then N_t is a completely semi essential fuzzy submodule of M .

Proposition(3.8) Let A be a fuzzy module of an R-module M and let N_1 and N_2 be two completely semi essential fuzzy submodules of A.Then $N_1 \cup N_2$ is a completely semi essential fuzzy submodule of A whenever $N_1 \subseteq N_2$ or $N_2\subseteq N_1$

Proof : If $N_1\subseteq N_2$

Let C be a completely prime fuzzy submodule of A such that :

$(N_1\cap N_2)\cap C=0_1$

$N_2\cap C=0_1$ (since $N_1\subseteq N_2$) .

$C=0_1$ (since N_2 completely semi essential fuzzy submodule of A) .

Similarly if $N_2\subseteq N_1$.

Proposition (3.9) Let A be a fuzzy module of an R-module M and let N_1 and N_2 be

two fuzzy submodules of A such that N_1 is a fuzzy submodule of N_2 if N_1 is a semi essential fuzzy submodule of A .Then N_2 is a completely semi essential fuzzy submodule of A .

Proof :

Let C be a completely prime fuzzy submodule of A such that $N_2 \cap C = 0_1$

$N_1 \cap C = 0_1$ (Since $N_1 \subseteq N_2$) .

C is a prime fuzzy submodule of A (By proposition (2.22))

$C = 0_1$ (Since N_1 semi-essential fuzzy submodule of A).

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الموديولات الضبابية الجزئية من النوع شبه جوهرية تماماً

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ملخص البحث:-

في هذا البحث ، قدمنا مفهوم الموديولات الضبابية الجزئية من النوع شبه جوهرية تماماً بالمعنى الاعتيادي والضبابي واعطينا بعض الخصائص عليها وكذلك درسنا علاقة هذا المفهوم بمفاهيم الموديولات الجوهرية وشبه الجوهرية بالمعنيين الضبابي والاعتيادي