CHROMATICITY OF WHEELS WITH MISSING THREE SPOKES

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Abstract. This paper show that every wheel of even order $n \geq 6$ with three missing spokes is chromatically unique.

1. Introduction

All graph consider here are simple graphs. For a graph G, Let $V(G)$ and $E(G)$ be the vertex set and a edge set of graph G, respectively. The order of G is denoted by $v(G)$, and the size of G by $e(G)$, i.e. $v(G) = |V(G)|$ and $e(G) = |E(G)|$. Let $P(G, \lambda)$ (or simply $P(G)$) denote the chromatic polynomial of graph G. Two graphs G and H are called chromatically equivalent if $P(G) = P(H)$, and G is called chromatically unique if $P(G) = P(H)$ implies H isomorphic to G for any graph H [9]. A wheel W_n is a graph obtained by taking the join of K_1 and the cycle C_{n-1}, edges which join K_1 to the vertices of C_{n-1} are called the spokes [2]. Let $\mathrm{W_n}$ be wheel of order n and let $\mathrm{W(n,k)}$ be the graph obtained from $\mathrm{W_n}$ by deleting all but k consecutive spokes, where $\mathrm{n \geq 4}$ and $1 \leq \mathrm{k} \leq \mathrm{n-1}$. Chia [2] showed that $\mathrm{W(n, n-2)}$ is chromatically unique for any even integers $\mathrm{n} \geq 6$. In [1], $W(5,3)$ was proved to be chromatically unique. Dong and Li, [5], proved that for any odd integer $n \geq 9, W(n, n-2)$ is chromatically unique, and just one graph, (shown in Fig.1(b)) is chromatically equivalent to $W(7,5)$, and is not isomorphic to it. It is easy to check that $W(4,1)$ and $W(5,2)$ are chromatically unique.

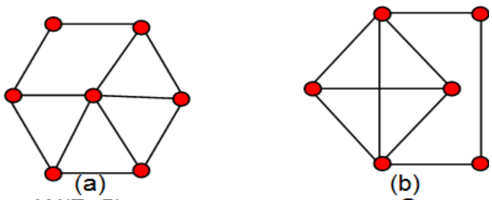


Fig. 1: Graphs W(7, 5) and G are chromatically equivalent but not isomorphic

2.2. Some Known Results

In this section, we introduce some known results, to be used in the following section.

Lemma 2.1.[6]: Let G and H be two chromatically equivalent graphs, then,

(a) $|V(G)| = |V(H)|, |E(G)| = |E(H)|$ and $t_1(G) = t_1(H)$, where $t_1(G)$ is the number of triangles in the graph G;

(b) $t_2(G) - 2t_3(G) = t_2(H) - 2t_3(H)$, where $t_2(G)$ and $t_3(G)$ are the number of cycles of order 4 without chords in G and the number of K_4's in G, respectively;

(c) G and H have the same number of components, and if G and H are connected, then G and H contain the same number of blocks; and (d) $\chi(G) = \chi(H)$.

Let G be a graph, for a subset S of $V(G)$, let $G[S]$ or $[S]$ denote the sub graph of G induced by $S, G[S]$ is called an induced sub graph of G. If S is a cut-set of G and $G[S]$ is complete, then S is called a complete cut-set of G.

Lemma 2.2. [8]: Let G and H be two chromatically equivalent graphs. If $\chi(G) = 3$ and $t_2(G) \geq 1$, then $3t_4(G)$ - $t_5(G)$ = $3t_4(H)$ - $t_5(H)$, where $t_4(G)$ and $t_5(G)$ are the number of induced wheel W_5' s in G and the number of induced pentagons in G, respectively. For a cycle C of G, if $|C| \geq 4$ and $G[C] \cong C$, (i.e., every two non-consecutive vertices of C are not adjacent), then C is called a chordless cycle of G. G is called a chordal graph if G contains no chordless cycles. For a vertex x of G, x is called a simplicial vertex of G if $d(x) = 0$ or $G\ [N_x]$ is complete.

Lemma 2.3.[3]: If G be a chordal graph, but not complete, then G contains two non-adjacent simplicial vertices.

Lemma 2.4.[4]: Let G be a connected graph of order n and x be a vertex of G with $d(x) < n - 1$. *Let* $S_1, S_2, \ldots, S_m$, where $m \geq 1$, be the vertex set of the all components of $G - x - N_x$, then, no chordless cycle of G contains x if and only if $G[N_x \cap N_{S_i}]$ is complete for every integer i with $1 \leq i \leq m$, where N_{S_i} is the set of all vertices of G adjacent to some vertices in S_i.

Corollary [7] : Let G be a connected graph of order n. If G contains no complete cut-sets, then, for any vertex $x \in V(G)$ with $d(x) < n - 1$, there exists a chordless cycle C of G, such that, $x \in V(C)$.

Graph G is called a forest if G contains no cycles, and G is called a tree if G is connected and contains no cycles. Obviously, a tree is also a forest. In what follows in this section, let G be a graph with chromatic number 3, and let A_1, A_2 *and* A_3 be the colour classes of a 3-colouring of G. Define $G_{i,j}$ to be $G[A_i \cup A_j]$ for all i and j with $1 \leq i < j \leq 3$

Lemma 2.5. [5]: If $G_{1,2}$, $G_{1,3}$ and $G_{2,3}$ are forests, and $d(x) = 0$ or $G[N_x]$ is a tree for every $x \in A_3$, then G is a chordal graph.

3. The Chromatic Uniqueness of $W(n, n-4)$ When $n \geq 6$ is Even.

In this section We use similar way of [7] to prove $W(n, n-4)$ is chromatically uniqueness for even $n \geq 6$. From now on, let G be a graph such that $P(G,\lambda) = P(W(n, n-4), \lambda)$ where $n \geq 6$, obviously $\chi(G) = 3$. Let A_1, A_2 *and* A_3 be the color classes of a 3-colouring of G and define $G_{i,j}$ to be $G[A_i \cup A_j]$ for all integer i and j *with* $1 \leq i < j \leq 3$. It is found that

$$P(G,\lambda) = P(Wn,\lambda) + 3P(W(n-1),\lambda) + 4P(W(n-2),\lambda) + 2P(W(n-3),\lambda) + P(W(n-4),\lambda)$$

$$P(W_n,\lambda) = \lambda[(\lambda-2)^{n-1} + (-1)^{n-1}(\lambda-2)]$$

$$3\,P(W(n-1),\lambda) = 3\lambda[(\lambda-2)^{n-2} + (-1)^{n-2}(\lambda-2)]$$

$$4\,P(W(n-2),\lambda) = 4\lambda\,[(\lambda-2)^{n-3} + (-1)^{n-3}(\lambda-2)]$$

$2\,P(W(n-3),\lambda) = 2\lambda[(\lambda-2)^{n-4} + (-1)^{n-4}(\lambda-2)]$

$P(W(n-4),\lambda) = \lambda[(\lambda-2)^{n-5} + (-1)^{n-5}(\lambda-2)]$

$P(G,\lambda) = \lambda(\lambda-2)^{n-5}[(\lambda-2)^4 + 3(\lambda-2)^3 + 4(\lambda-2)^2 + 2(\lambda-2) + 1] + (-1)^{n-5}\lambda(\lambda-2)[(-1)^4 + 3(-1)^3 + 4(-1)^2 + 2(-1) + 1]$

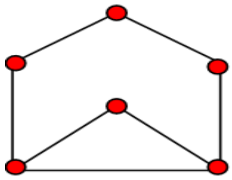
$=\lambda(\lambda-2)^{n-5}[(-1)^{n-5}(\lambda-2)^{-(n-6)} + (\lambda-2)^4 + 3(\lambda-2)^3 + 4(\lambda-2)^2 + 2(\lambda-2) + 1]$

$P(G,\lambda) = \lambda(\lambda-2)^{n-5}[(\lambda-2)^{-(n-6)} + (\lambda-2)^4 + 3(\lambda-2)^3 + 4(\lambda-2)^2 + 2(\lambda-2) + 1], n$ is odd.

$P(G,\lambda) = \lambda(\lambda-2)^{n-5}[-(\lambda-2)^{-(n-6)} + (\lambda-2)^4 + 3(\lambda-2)^3 + 4(\lambda-2)^2 + 2(\lambda-2) + 1]$, n is even.

Lemma 3.1: $W(6,2)$ is chromatically unique.

Proof: $W(6,2)$ is graph of K_2- gluing of complete graph k_3 and a cycle C_5, by Theorem (3.6.2 [6]), $W(6,2)$ is chromatical unique.See (Fig. 2) ■



W(6, 2)
Fig. 2: Graph W(6, 2) is unique

By lemmas 2.1 and 2.2 the following result is obtained

Lemma 3.2: G has the following properties:

(a). G has n vertices, $2n-5$ edges and $n-5$ triangles,

(b). $\chi(G) = 3$ and G is 2 -connected,

(c). $P(G,\lambda)$ is devisable by neither $(\lambda-1)^2$ nor $(\lambda-2)^2$

(d). $t_2(G) = 0, t_3(G) = o, t_4(G) = 0, t_5(G) = 0, t_6(G) = 0,\ t_7(G) = 1$, where $t_6(G) = K_5$, $t_7(G) = C_6$.

(e) There are five ways when n is even and six when n is odd to separate the vertex set of G three independent subsets, since

$$S(G) = \frac{P(G,\ 3)}{3!} = \begin{cases} 5, & where\ n\ is\ even \\ 6, & other\ wise \end{cases}$$

Corollary: Let S be a complete cut-set of G, and G_1 and G_2 be two induced proper subgraphs of G such that $G_1 \cup G_2 = G$ and $V(G_1) \cap V(G_2) = S$, then G_1 and G_2 are not chordal graphs.

Poof: Suppose G_1 is a chordal graph. By lemma 2.3 there is a vertex y in G_1 such that y is a simplicial vertex of G. Thus $P(G,\lambda) = (\lambda - d(y))\,P(G-y,\lambda)$. Since $\chi(G) = 3$ and G is 2-connected $d(y) = 2$. Since $t_1(G) > 1$, $\chi(G-y) = 3$. So $P(G,\lambda)$ can be divisible by $(\lambda-2)^2$, contrary to lemma 3.2.Thus G_1 is not a chordal graph, and similarly, G_2 is also a non-chordal graph. ■

Lemma 3.3: Among the three subgraphs $G_{1,2}, G_{1,3}\ and\ G_{2,3}$, two are forests with two components and the other one is tree. Without loss of generality, suppose that $G_{1,2}$ and $G_{1,3}$ are forests with two components and thus $G_{2,3}$ is tree.

Proof: Since $s(G) = 5$, there is at most two sub graphs $G_{i,j}$, which are not connected, where

$1 \leq i < j \leq 3$ without loss of generality, suppose that $G_{1,2}$ and $G_{1,3}$ are not connected. Thus $G_{2,3}$ is possibly not connected. See Fig. 3. So we have

$$|E(G_{1,2})| \geq |V(G_{1,2})| - 2$$

$$|E(G_{1,3})| \geq |V(G_{1,3})| - 2$$

$$|E(G_{2,3})| \geq |V(G_{2,3})| - 1$$

and therefor $|E(G)| = \sum E(G_{i,j}) \geq 2|V(G)| - 5 = 2n - 5$

Since $|E(G)| = 2n - 5$. We have the following result

$$|E(G_{1,2})| \geq |V(G_{1,2})| - 2$$

$$|E(G_{1,3})| \geq |VG_{1,3})| - 2 \qquad (1)$$

$$|E(G_{2,3})| \geq |V(G_{2,3})| - 1$$

And thus $G_{1,2}$ and $G_{1,3}$ are forests with two components, and $G_{2,3}$ is tree.■

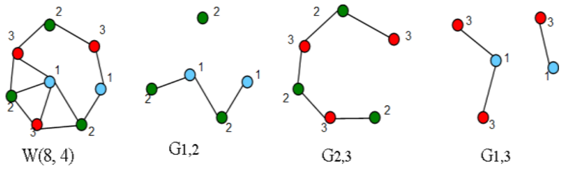


Fig. 3: Three supgraphs of 3-colouring of W(8, 4)

let $1 + \rho(x)$ be the number of components of H, and let $\rho(x) = \rho(H[N_x])$ for any $x \in V(H)$. Then $\rho(x) \geq -1$ for any $x \in V(H)$. Therefore, more $\rho(x) = -1$ iff x is an isolated vertex, and $\rho(x) = 0$ iff N_x is connected. Since G is 2-connected, $\rho(x) \geq 0$ for any $x \in V(G)$. $G[N_x]$ is subgraph of $G - A_i$ if x is in $A_i, 1 \leq i \leq 3$, thus by Lemma 3.3 for any vertex $x \in V(G), G[N_x]$ is a forest, and $G[N_x]$ is tree if and only if $\rho(x) = 0$. We calculate the number of triangles of G in the following way

$$t_1(G) = \sum_{x \in A_1}(d(x) - 1 - \rho(x))$$

$$= \sum_{x \in A_1} d(x) - |A_1| - \sum_{x \in A_1} \rho(x)$$

$$= |E(G)| - |E(G_{2,3})| - |A_1| - \sum_{x \in A_1} \rho(x)$$

$$= |E(G)| - |E(G_{2,3})| - \{|V(G)| - |V(G_{2,3})|\} - \sum_{x \in A_1} \rho(x)$$

$$= |E(G)| - |V(G)| + \{|V(G_{2,3})| - |E(G_{2,3})|\} - \sum_{x \in A_1} \rho(x)$$

$$= |E(G)| - |V(G)| + 1 - G_{2,3}$$

$$n - 5 = 2n - 5 - n + 1 - \sum_{x \in A_1} \rho(x)$$

$$\sum_{x \in A_1} \rho(x) = 1 \qquad (2)$$

$$t_1(G) = \sum_{x \in A_2}(d(x) - 1 - \rho(x))$$

$$= \sum_{x \in A_2} d(x) - |A_2| - \sum_{x \in A_2} \rho(x)$$

$$= |E(G)| - |E(G_{1,3})| - |A_2| - \sum_{x \in A_2} \rho(x)$$

$$= |E(G)| - |E(G_{1,3})| - \{|V(G)| - |V(G_{1,3})|\} - \sum_{x \in A_2} \rho(x)$$

$$= |E(G)| - |V(G)| + \{|V(G_{1,3})| - |E(G_{1,3})|\} - \sum_{x \in A_2} \rho(x)$$

$$= |E(G)| - |V(G)| + 2 - \sum_{x \in A_2} \rho(x)$$

$$n - 5 = 2n - 5 - n + 2 - \sum_{x \in A_2} \rho(x)$$

$$\sum_{x \in A_2} \rho(x) = 2 \qquad (3)$$

$$t_1(G) = \sum_{x \in A_3}(d(x) - 1 - \rho(x))$$

$= \sum_{x\in A_3} d(x) - |A_3| - \sum_{x\in A_3} \rho(x)$
$= |E(G)| - |E(G_{1,2})| - |A_3| - \sum_{x\in A_3} \rho(x)$
$= |E(G)| - |E(G_{1,2})| - \{ |V(G)| - |V(G_{1,2})|\} - \sum_{x\in A_3} \rho(x)$

$= |E(G) - |V(G)| + \{|V(G_{1,2})| - |E(G_{1,2})| - \sum_{x\in A_3} \rho(x)$

$=|E(G)| - |V(G)| + 2 - \sum_{x\in A_3} \rho(x))$

$n - 5 = 2n - 5 - n + 2 - \sum_{x\in A_3} \rho(x),$

$\sum_{x\in A_3} \rho(x) = 2$ (4)

Lemma3.4:There exist five different vertices $x_1 \in A_1$, $x_2^1, x_2^2 \in A_2$, $x_3^1, x_3^2 \in A_3$ that for any x

$$\rho(x) = \begin{cases} 1, & where\ x = x_1, x_2^1, x_2^2, x_3^1, x_3^2 \\ 0, & otherwise \end{cases} \quad (5)$$

Proof: By (2), (5) is obtained for $x_1 \in A_1$, if there is a vertex $x_2 \in A_2$such that $\rho(x_2) \geq 2$, then by (3) $\rho(x_2) = 2$ and $\rho(x) = 0$ for any $x \in \{A_2\}\backslash\{x_2\}$. Thus $G - \{x_2\}$ is not connected, i. e $\{x_2\}$ is a cut vertex of G, contrary to lemma 3.2 (b). So there exist two different vertex, x_2^1, x_2^2 such that $\rho(x_2^1) = 1$, $\rho(x_2^2) = 1$ and $\rho(x) = 0$ for any $x \in \{A_2\}\backslash\{x_2^1, x_2^2\}$. Similarly $\rho(x_3^1) = 1$, $\rho(x_3^2) = 1$ and $\rho(x) = 0$ for any $x \in \{A_3\}\backslash\{x_3^1, x_3^2\}$.■

Lemma 3.5: Any cordless cycle of G contains the vertices, $x_1, x_2^1, x_2^2, x_3^1\ and\ x_3^2$.

Proof: By lemma 2.5 and lemma 3.4 $G - x_1$ is chordal graph. So any chordless cycle of G contains x_1. Since $G_{1,2}$ and $G_{1,3}$ are not connected with two components and $\rho(x) = 0$ for any $x \in \{A_2\}\backslash\{x_2^1, x_2^2\}$, similarly $\rho(x) = 0$ for any $x \in \{A_3\}\backslash\{x_3^1, x_3^2\}$. Therefore $G - x_2^1 - x_2^2$ is not connected to two components. Let S_1 and S_2 denote the vertex sets of the two components, and $G_i = G[S_i \cup \{x_2^1, x_2^2\}]$, $i = 1, 2$. For any vertex $x \in V(G_1) \cap A_2$, $G_1 [N_x]$ is tree. By lemma 2.5, G_1 is a chordal graph. Therefor, any chcordless cycle of G contains x_2^1, x_2^2, similarly, any chordless cycle of G contains x_3^1, x_3^2. ■

Lemma 3.6: G contain no complete cut-set.

Proof: Assume that A is complete cut-set of G. Let G_1, G_2 be two induced proper subgraphs of G such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 =$ G[A]. By corollary to lemma 3.2 , $G_1$and G_2 are not chordal graphs. Since $G[A]$ is complete, any chordal cycle of G is in G_1or in G_2. Then, by lemma 3.5, the five vertices x_1, x_2^1, x_2^2, x_3^1 and x_3^2 must be in A, contrary to $G[A]$ being complete. ■

Lemma 3.7: G contains just one vertex induced sixth in G and $x_3^1x_2^1x_1x_3^2x_2^2$ or $x_3^1x_2^1x_3^2x_1x_2^2$ or $x_2^1x_3^1x_2^2x_1x_3^2$ or $x_3^1x_2^1x_3^2x_2^2x_1$or $x_1x_3^1x_2^1x_3^2x_2^2$, or $x_3^1x_2^1\ x_1x_2^2x_3^2$, or $x_2^1x_3^1\ x_1x_3^2x_2^2$, is a path in the sixth

proof: By lemma 3.1, there is exactly one induced sixth in G. By lemma 3.4 , x_1, x_2^1, x_2^2, x_3^1 and x_3^2 are in the sixth. Then five vertices induced path. Since x_2^1 and x_2^2 are in A_2, the two vertices are not consecutive in the path, so that x_3^1 and x_3^2 are in A_3, the two

vertices are not consecutive in the path. This path can be expressed in seven different ways, $x_3^1x_2^1x_1x_3^2x_2^2$ or $x_3^1x_2^1x_3^2x_1x_2^2$ or $x_2^1x_3^1x_2^2x_1x_3^2$ or $x_3^1x_2^1x_3^2\,x_2^2x_1$ or $x_1x_3^1x_2^1x_3^2x_2^2$ or $x_3^1x_2^1\,x_1x_2^2x_3^2$ or $x_2^1x_3^1\,x_1x_3^2x_2^2$, see Fig. 4

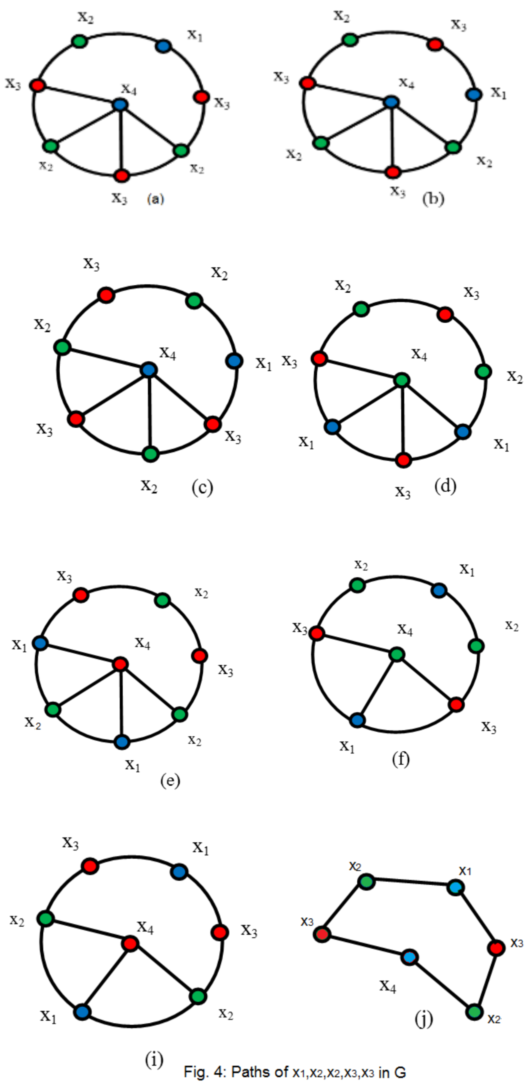


Fig. 4: Paths of x1,x2,x2,x3,x3 in G

Lemma 3.8: If $x_3^1x_2^1x_1x_3^2x_2^2$ or $x_3^1x_2^1x_3^2x_1x_2^2$ or $x_2^1x_3^1x_2^2x_1x_3^2$ or $x_3^1x_2^1x_3^2\,x_2^2x_1$ or $x_1x_3^1x_2^1x_3^2x_2^2$ is path in G, then it is isomorphic to $W(n,n-4)$.

Proof: Without loss of generality let $x_3^1x_2^1x_1x_3^2x_2^2$ be path in G. Let $x_4x_3^1x_2^1x_1x_3^2x_2^2x_4$ be the six in G, (as show in Fig. 4(j), where $x_4 \in A_1$), by lemma 3.4 $\rho(x_4) = 0$, i.e., $G[N_{x_4}]$ is a tree. So there exists the unique path $P(x_3^1, x_2^2)$ in $G_{2,3}$ connecting x_3^1 and x_2^2, and x_4 is adjacent to each vertex in the path. Thus $p(x_3^1, x_2^2)$ dose not contain x_2^1 or x_3^2 or x_1. Let G_0 be subgraph $G\left[V\left(P(x_3^1, x_2^2)\right) \cup \{x_2^1, x_3^2, x_1, x_4\}\right]$,
by lemma 3.3 G_0 isomorphic to $W(m, m-4)$ for some even integer m, (as show in Fig.4 (a,b,c,d,e)). Suppose C is cordless cycle of G and C is not in G_0. By lemma 3.5, C contai x_3^1, x_2^1, x_1, x_2^2 and x_3^2. If x_4 is not adjacent to some vertices on the path $C - x_2^1 - x_1 - x_3^2$, then there exists a chordless cycle in the subgraph induced by $\{x_4\} \cup V(C - x_2^1 - x_3^2 - x_1)$, which dose not contain x_2^1, x_3^2, x_1 contrary to lemma 3.5. If x_4 is adjacent to each vertex in $(C - x_2^1 - x_3^2 - x_1)$, then the sub graph of G induced by $V(C - x_2^1 - x_3^2 - x_1) \cup V(P(x_3^1, x_2^2)$ is asubgraph of $G_{2,3}$, which contains cycles, contrary to $G_{2,3}$ being a tree. Therefor, any chordless cycle of G is in the subgraph G_0. Let x be any vertex of G. If $d(x) = n-1$ and $x \in A_i$, then $|A_i| = 1$. But, from the subgraph G_0, we know $|A_i| \geq 2$ for any $i = 1,2,3$. Thus $d(x) < n-1$ for any $x \in V(G)$, by lemma 3.6 and the corollary to lemma 2.4, x is contained in some chordless cycle of G, and thus $x \in V(G_0)$, which implies that $V(G) = V(G_0)$. So $G \cong G_0$, i. e ,$G = W(m, m-4)$ for some even integer m. Hence $m = n$ and $G \cong W(n, n-4)$.■

Lemma 3.9: G dose not contain any one of the two paths

$x_3^1 x_2^1 x_1 x_2^2 x_3^2 \quad or \;\; x_2^1 x_3^1 x_1 x_3^2 x_2^2.$

Proof: If G contains one of the five path, we can show, in the same way, as in lemma 3.8, that $G \cong W(m, m-4)$ for some odd integer m is a contradiction. See Fig.4 (f, i). ■

Theorem 1: For any even integer $n \geq 6$, the graph $W(n, n-4)$ is chromatically unique

References

[1] C. Y. Chao and E. G. Whitehead Jr., Chromatically unique graphs, *Discrete Math.* 26 (1979) 171-17.

[2] G. L. Chia, The chromaticity of wheels with missing spoke, *Discrete Math*. 82 (1990) 209- 212.

[3] G. A. Dirac, On rigid circuit graphs, *Abh. Math. Sem. Univ. Hamburg* 25 (1967) 71-76.

[4] F. M. Dong, On the chromatic uniqueness of generalized wheel graphs, *Math. Res. Exposition* 10 (1990) 76 - 83 (in Chinese).

[5] F. M. Dong and Y.P. Liu and K. M. Koh, Almost all wheels with one missing spoke are chromatically unique.

[6] F. M. Dong, K. M. Koh and K. L. Teo, Chromatic polynomials and chromaticity of graphs, *World scientific publishing co. Pte. Ltd*, 2005.

[7] F. M. Dong and P. Liu, All wheels with two missing consecutive spokes are chromatically unique, *Discrete Math*. 184 (1998) 71-85.

[8] E. J. Farrell, On chromatic coefficients, *Discrete Math.* 29 (1980) 257-264. MR81d:05029.

[9] K.M. Koh, K. L.Teo, The search for chromatically unique graphs, *Discrete Mathematics* 172 (1997) 59-78.