Solution of Singular Fuzzy Fraction order Control Systems via Drazin inverse

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Abstract:
In this paper , we investigate solution for singular fuzzy fractional order control system , which in turn gives us two subsystem by drazin inverse . One of these subsystems contain fuzzy fractional order control system used fuzzy laplace transform as well as lower function and upper function in order to get the solution .

Key words :singular , Drazin inverse , fractional , control system , fuzzy solution .

1. Introduction

Singular systems have received much attention by singular model can preserve the structure of practical systems better than regular ones. Singular systems are also referred as descriptor, semi-state, linear system models in many applications such as electrical networks, aircraft dynamics, neutral delay systems, chemical, thermal and diffusion processes, large-scale systems, interconnected systems, economics, optimization problems, feedback systems, robotics, biology, etc [10,16]."Fractional calculus, which was introduced in the early 17th century, deals with integration and derivatives of arbitrary noninteger order". In recent years, it has been reported in many areas such as electrical circuit, population models, epidemiology models,[2].A lot of researchs has been focused on the application of fractional calculus, and such application is in the modeling of many physical and chemical processes as well as in engineering. It has been found that the behavior of many physical systems can be properly described by using the fractional order system theory. "Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. The advantages or the real objects of the fractional order systems are that have more degrees of freedom in the model and that a “memory” is included in the model". [3–6,11].Control systems are ubiquitous. They appear in homes, cars, industry and in systems for communication and transport, just to give a few examples. Control is increasingly becoming mission critical, processes will fail if the control does not work. Control has been important for design of experimental equipment and instrumentation used in basic sciences and will be even more so in the future. "Principles of control have an impact on such fields as economics, biology, and medicine. Control, like many other branches of engineering science, has developed in the same pattern as natural science. Although there are strong similarities between natural science and engineering science is important to realize that there are some fundamental differences. The inspiration for natural science is to understand phenomena in nature".[12] ."The concept of fuzzy numbers and fuzzy arithmetic operations were first introduced by Zadeh [8, 19]. The importance of the introduced notion of fuzzy set was realized and has successfully been applied in the branches of science and technology. Recently fuzzy set theory has been applied in pure mathematics by" [18].

In this paper is consists of three sections . section two deals with some of the basic mathematics concepts and principles of singular fractional order Control system with the fuzzy numbers finally section three the singular fuzzy fractional order control system is discussed .

2. Preliminaries

Definition (2.1) , [14]:
"The function defined by

$${}_0^cD_t^{\alpha}f(t)=\frac{1}{\Gamma(q-\alpha)}\int_0^t\frac{f^{(q)}(\tau)}{(t-\tau)^{\alpha+1-q}}d\tau\ , \qquad (1a)$$

is called the Caputo fractional derivative-integral, where

$q-1<\alpha<q, q\in\mathbb{N}$. So , for n=1 $0<\alpha<1$, the relations above take the following"

$${}_0^cD_x^{\alpha}f(x)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{\frac{d}{dt}f(\tau)}{(t-\tau)^{\alpha}}d\tau \qquad (1b)$$

Definition (2.2) , [14]:

"The function defined by

$${}_0I_t^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau)d\tau\ , \qquad (2)$$

is called the Riemann-Liouville fractional integral, where $\alpha\in\mathbb{R}_+$ is the order of integral.

The Mittag-Leffler function is a generalization of the exponential function e^{s_it} and it plays an important role in the solution of singular fractional order differential equations"

Definition (2.3), [14]:

"A function of the complex variable z defined by $E_{\alpha,1}(z)=\sum_{k=o}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}\ ,\alpha>0$ (3)

is called the one parameter Mittag-Leffler function. Sometime it is denoted by" $E_{\alpha,1}(z)=E_{\alpha}(z)$.

Laplace transformation is fundamental tool in system and control engineering. For this reason, we will give here the equation of these transform for defined fractional order operator.

Lemma (2.1), [9]:

1- ${}_0^cD_t^{\alpha}f(t)=s^{\alpha}F(s)-\sum_{j=0}^{q-1}s^{\alpha-j-1}f^{(j)}(0)$,
$q-1<\alpha\le q\ ,\ q\in\mathbb{N}$.-

2- $\mathcal{L}[{}_0^cD_t^{\alpha}f(t)]=s^{\alpha}F(s)-s^{\alpha-1}f(0), 0<\alpha\le 1$.

3- Laplace transformation of convolution,

$$f(t)*g(t)=\int_0^t f(t-\tau)g(\tau)d\tau$$

$=\int_0^t f(\tau)g(t-\tau)d\tau$, then

$$\mathcal{L}[f(t)*g(t)]=F(s)G(s)\ .$$

Definition (2.4) (Control), [17]:

"Control means measuring the value of the controlled variable of the system and applying the manipulated variable to the system to correct or limit deviation of the measured value from a desired value".

Definition (2.5) (index of the matrix), [13]:

"The smallest nonnegativeinteger qsatisfying $\text{rank}\bar{E}^q=\text{rank}\bar{E}^{q+1}$,$\bar{E}\in\mathbb{R}^{n\times n}$is called the index of the matrix" $\bar{E}\in\mathbb{R}^{n\times n}$.

Definition (2.6) (Drazin inverse), [7] , [13]:

"A matrix $\bar{E}^D$ is called theDrazin inverse of $\bar{E}\in\mathbb{R}^{n\times n}$ if it satisfies the following conditions

i) $\bar{E}\bar{E}^D=\bar{E}^D\bar{E}$

ii) $\bar{E}^D\bar{E}\,\bar{E}^D=\bar{E}^D$

iii) $\bar{E}^D\bar{E}^{q+1}=\bar{E}^q$ (4)

where q is the index of $\bar{E}$defined by (4).

The Drazin inverse $\bar{E}^D$of a square matrix $\bar{E}$ always exists and is unique".

Definition (2.7) (Fuzzy number), [20]

"A fuzzy number is a mapping $v:R\to[0,1]$ with the following properties:

i) v is upper semi continuous
ii) $v(x)=0$ outside some interval [a ,d]
iii) There are real numbers b and c , $a\le b\le c\le d$, for which
 i) $v(x)$ is monotonically increasing on [a ,b]
 ii) $v(x)$ is monotonically decreasing on [c ,d]
 iii) $v(x)=1$, $b\le x\le c$."

Definition (2.8), [1]

"A fuzzy number v in parametric form is a pair of functions($\underline{v}(r)$,$\overline{v}(r)$), $0\le r\le 1$, which satisfy the following requirements

i)$\underline{v}(r)$ is a bounded left continuous non-decreasing function over [0,1] .

ii)$\overline{v}(r)$ is a bounded left continuous non-increasing function over [0,1] .

iii)$\underline{v}(r)\le\overline{v}(r)$, $0\le r\le 1$.

The set of fuzzy numbers is denoted by F. For arbitrary fuzzy $\tilde{x}=(\underline{x}(r),\overline{x}(r))$, $\tilde{y}=(\underline{y}(r),\overline{y}(r))$ and∈ R , we may define the addition and the scalar

multiplication of fuzzy numbers by using the extension principleas" [1]:

1) $\tilde{x}+\tilde{y}=(\underline{x}(r)+\underline{y}(r),\overline{x}(r)+\overline{y}(r))$,

2) $k\tilde{x}=\begin{cases}(k\underline{x},k\overline{x}), k\geq 0\\ (k\overline{x},k\underline{x}), k<0\end{cases}$ (5)

Lemma (2.2), [15]:

"Consider the nonlinear fractional order system $D^{\propto}x(t)=f(t,x(t))$

Suppose $\propto\in(0,1], G>0$ and F is fuzzy real number. Then, the fractional fuzzy differential is $D^{\propto}x(t)=$

$f(t,x(t))$, $t\in(0,G], x_0\in F$ (6)

Where $f(0,G]\times F\to F$ is continuous in the case of $f(0,G]\times R\to R$ and $x_0\in R$, $\lim_{n\to\infty} t^{1-\propto}x(t)=x_0\in F, f(t,x(t))=\lambda x+g(t,x))$, $g:(0,G]\times F\to F$,is cont , then the solution" is $x(t)=\Gamma(\alpha)t^{\propto-1}E_{\alpha,\alpha}(\lambda t^{\propto})x_0$

$$+\int_0^t((t-s)^{\propto-1}E_{\alpha,\alpha}(\lambda(t-s)^{\propto})g(s,x(s)))ds$$

Where $E_{\alpha,\alpha}(z)=\sum_{k=0}^{\infty}\frac{z^{k\alpha}}{\Gamma(\alpha(k+1))}$

3.singular fuzzy fractional differential order control system

Consider the following singular fractional order control system

$E[D^{\propto}x(t)+g(t,x(t))]=Ax(t)+Bu(t)$, $t\in[0,G], G>0$ (7)

$x(0)=x_0=\begin{bmatrix}x_{1\ 0,r}\\ x_{2\ 0,r}\end{bmatrix}\in F$,$0<\propto\leq 1$,F is fuzzy real number where $D^{\propto}x(t)$ is Caputo fractional derivative of $x(t)$, $E\in\mathbb{R}^{n\times n}$ be a singular matrix, $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, $x(t)\in\mathbb{R}^n$ is the state vector, $u(t)\in\mathbb{R}^m$ is the control input, the $g(t,x(t))$ be function define in condition(vi) (8)with the following :

i) $\det(Es-A)\neq 0$. For some $s\in C$ is the complex number

ii) $\overline{E}=P\begin{bmatrix}J & 0\\ 0 & N\end{bmatrix}P^{-1}$,

$\overline{E}^D=P\begin{bmatrix}J^{-1} & 0\\ 0 & 0\end{bmatrix}P^{-1}$, $\det P\neq 0$, $J\in\mathbb{R}^{r_1\times r_1}$, $\overline{E}$define in eq (8) , N is nilpotent matrix whose nilpotent index is k such that $N^K=0$,

iii) $P=\begin{bmatrix}P_1 & P_2\\ P_3 & P_4\end{bmatrix}$,

iv) $\overline{A}=[Ec-A]^{-1}A=PA$, $A=\begin{bmatrix}A_1 & A_2\\ A_3 & A_4\end{bmatrix}$

v) $\overline{B}=[Ec-A]^{-1}B=PB$, $B=\begin{bmatrix}B_1\\ B_2\end{bmatrix}$ where $B_1\in\mathbb{R}^{r_1\times m}$,$B_2\in\mathbb{R}^{n-r_1\times m}$

vi) $g(t,x)=g(t,x_1(t),x_2(t))$

$=\begin{bmatrix}g_1(x_1(t),x_2(t)))\\ g_2(x_1(t),x_2(t))\end{bmatrix}=$

$\begin{bmatrix}g_1(x_1(t),x_2(t)))\\ 0\end{bmatrix}$, $g:(0,G]\times F\to F$,is continuous in the case of $g(0,G]\times R\to R$(8)

Now ,multiple both sides by $[Ec-A]^{-1}$ for (7) , we get

$$\overline{E}[D^{\propto}x(t)+g(t,x(t))]=\overline{A}x(t)+\overline{B}u(t) \quad (9)$$

Where $\overline{E}=[Ec-A]^{-1}E, \overline{A}=[Ec-A]^{-1}A$ and $\overline{B}=[Ec-A]^{-1}B$

Theorem (3.1):

The general solution to the system (9) is given by

$$\varphi=\begin{bmatrix}\underline{x_1}(t,r)\ ,\overline{x_1}(t,r)\\ \underline{x_2}(t,r)\ ,\overline{x_2}(t,r)\end{bmatrix}$$

$$=\begin{pmatrix}(E_{\alpha,1}(\overline{A}_{11}t^{\alpha})\underline{x}_{1\ 0,r}\\ +\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(\overline{A}_{11}(t-s)^{\alpha})\widehat{B}_{11}u(s)ds\\ +\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(\overline{A}_{11}(t-s)^{\alpha})g_1\left(\underline{x_1}(s,r),\underline{x_2}(s,r)\right)ds\ , E_{\alpha,\alpha}(\overline{A}_{11}t^{\alpha})\overline{x}_{1\ 0,r}\\ +\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(\overline{A}_{11}(t-s)^{\alpha})\widehat{B}_{11}u(s)ds\\ +\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(\overline{A}_{11}(t-s)^{\alpha})g_1(\overline{x_1}(s,r),\overline{x_2}(s,r))ds)\\ \\ (-(P_3J^{-1}A_2)^{-1}P_3J^{-1}A_1\underline{x}_1(t)-(P_3J^{-1}A_2)^{-1}P_3J^{-1}B_2u(t),\\ -(P_3J^{-1}A_2)^{-1}P_3J^{-1}A_1\overline{x}_1(t)-(P_3J^{-1}A_2)^{-1}P_3J^{-1}B_2u(t))\end{pmatrix}$$

Where $E_{\alpha,1}(\overline{A}_{11}t^{\alpha})=\sum_{k=0}^{\infty}\frac{\overline{A}_{11}{}^{k}t^{k\propto}}{\Gamma(k\propto+1)}$,

$E_{\alpha,\alpha}(\overline{A}_{11}(t-s)^{\alpha})=\sum_{k=0}^{\infty}\frac{\overline{A}_{11}{}^{k}(t-s)^{k\propto}}{\Gamma(k\propto+\alpha)}$

Proof:

both sides by Drazin inverse of the matric $\overline{E}$. for (9) , we obtain

$$\overline{E}^D\overline{E}[D^{\propto}x(t)+g(t,x(t))]=\overline{E}^D\overline{A}x(t)+\overline{E}^D\overline{B}u(t)$$

By conditions of eq (8) , we get

$$P\begin{bmatrix} J^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1} P \begin{bmatrix} J & 0 \\ 0 & N \end{bmatrix} P^{-1}\left(\begin{bmatrix} D^{\propto}x_1(t) \\ D^{\propto}x_2(t) \end{bmatrix} + \begin{bmatrix} g_1(x_1(t), x_2(t))) \\ 0 \end{bmatrix}\right)$$

$$= P\begin{bmatrix} J^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1} P \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + P\begin{bmatrix} J^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1} P\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t)$$

Then ,

$$\begin{bmatrix} I_{r_1} & 0 \\ 0 & 0 \end{bmatrix}\left(\begin{bmatrix} D^{\propto}x_1(t) \\ D^{\propto}x_2(t) \end{bmatrix} + \begin{bmatrix} g_1(x_1(t), x_2(t))) \\ 0 \end{bmatrix}\right) = P\begin{bmatrix} J^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + P\begin{bmatrix} J^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t)$$

One can get,

$$\begin{bmatrix} I_{r_1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} D^{\propto}x_1(t) + g_1(x_1(t), x_2(t))) \\ D^{\propto}x_2(t) \end{bmatrix} = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}\begin{bmatrix} J^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}\begin{bmatrix} J^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t)$$

Thus,

$$\begin{bmatrix} I_{r_1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} D^{\propto}x_1(t) + g_1(x_1(t), x_2(t))) \\ D^{\propto}x_2(t) \end{bmatrix} = \begin{bmatrix} P_1J^{-1}A_1 & P_1J^{-1}A_2 \\ A_3 & A_4 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} P_1J^{-1} & 0 \\ P_3J^{-1} & 0 \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t)$$, We get ,

$$\begin{bmatrix} D^{\propto}x_1(t) + g_1(x_1(t), x_2(t))) \\ 0 \end{bmatrix} = \begin{bmatrix} P_1J^{-1}A_1x_1(t) + P_1J^{-1}A_2x_2(t) \\ P_3J^{-1}A_1x_1(t) + P_3J^{-1}A_2x_2(t) \end{bmatrix} + \begin{bmatrix} P_1J^{-1}B_1 \\ P_3J^{-1}B_2 \end{bmatrix} u(t)$$, We obtain ,

$$D^{\propto}x_1(t) + g_1(x_1(t), x_2(t))) = P_1J^{-1}A_1x_1(t) + P_1J^{-1}A_2x_2(t) + P_1J^{-1}B_1u(t) \quad (10a)$$

$$0 = P_3J^{-1}A_1x_1(t) + P_3J^{-1}A_2x_2(t) + P_3J^{-1}B_2u(t) \quad , (10b)$$

Now, $x_2(t)$ of equa(10b) ,

$$P_3J^{-1}A_2x_2(t) = -P_3J^{-1}A_1x_1(t) - P_3J^{-1}B_2u(t)$$

$$x_2(t) = -(P_3J^{-1}A_2)^{-1}P_3J^{-1}A_1x_1(t) - (P_3J^{-1}A_2)^{-1}P_3J^{-1}B_2u(t) \quad (11b)$$

Substituting (11b) in (10a), we obtain

$$D^{\propto}x_1(t) = P_1J^{-1}A_1x_1(t) + P_1J^{-1}A_2\begin{bmatrix} -(P_3J^{-1}A_2)^{-1}P_3J^{-1}A_1x_1(t) \\ -(P_3J^{-1}A_2)^{-1}P_3J^{-1}B_2u(t) \end{bmatrix} + P_1J^{-1}B_1u(t) - g_1(x_1(t), x_2(t)))$$

Hence,

$$D^{\propto}x_1(t) = P_1J^{-1}A_1x_1(t) - P_1J^{-1}A_2(P_3J^{-1}A_2)^{-1}P_3J^{-1}A_1x_1(t) - P_1J^{-1}A_2(P_3J^{-1}A_2)^{-1}P_3J^{-1}B_2u(t) + P_1J^{-1}B_1u(t) - g_1(x_1(t), x_2(t)))$$

follows,

$$D^{\propto}x_1(t) = [P_1J^{-1}A_1 - P_1J^{-1}A_2(P_3J^{-1}A_2)^{-1}P_3J^{-1}A_1]x_1(t) + [P_1J^{-1}B_1 - P_1J^{-1}A_2(P_3J^{-1}A_2)^{-1}P_3J^{-1}B_2]u(t) - g_1(x_1(t), x_2(t))) \quad (12a)$$

Now , using fuzzy fractional based on its lower and upper functions of the eq (12a),

$$\left(D^{\propto}\underline{x_1}(t), D^{\propto}\overline{x_1}(t)\right) = (P_1J^{-1}A_1 - P_1J^{-1}A_2(P_3J^{-1}A_2)^{-1}P_3J^{-1}A_1)\left(\underline{x_1}(t), \overline{x_1}(t)\right) + [P_1J^{-1}B_1 - P_1J^{-1}A_2(P_3J^{-1}A_2)^{-1}P_3J^{-1}B_2]u(t) - \left(g_1\left(\underline{x_1}(t), \underline{x_2}(t)\right), g_1\left(\overline{x_1}(t), \overline{x_2}(t)\right)\right) \quad (13a)$$

And the fuzzy initial condition ,

$$x(0,r) = \left(\underline{x}(0,r), \overline{x}(0,r)\right) = \left(\underline{x}_{\,0,r}, \overline{x}_{\,0,r}\right) = \left(\begin{bmatrix} \underline{x}_{1\ 0,r} \\ \underline{x}_{2\ 0,r} \end{bmatrix}, \begin{bmatrix} \overline{x}_{1\ 0,r} \\ \overline{x}_{2\ 0,r} \end{bmatrix}\right) \quad (13b)$$

now, solving (12) by using fuzzy Laplace transformation by two cases, depended on lower and upper functions:

The equation (13a) with the lower function is

$$(D^{\propto}\underline{x_1}(t,r)) = (P_1J^{-1}A_1 - P_1J^{-1}A_2(P_3J^{-1}A_2)^{-1}P_3J^{-1}A_1\underline{x_1}(t,r) + [P_1J^{-1}B_1 - P_1J^{-1}A_2(P_3J^{-1}A_2)^{-1}P_3J^{-1}B_2]u(t) - g_1\left(\underline{x_1}(t,r), \underline{x_2}(t,r)\right) \quad (14)$$

The equation(13a) with the upper function is

$$D^{\propto}\overline{x_1}(t,r) = (P_1J^{-1}A_1 - P_1J^{-1}A_2(P_3J^{-1}A_2)^{-1}P_3J^{-1}A_1\overline{x_1}(t,r) + [P_1J^{-1}\widehat{B}_1 - P_1J^{-1}A_2(P_3J^{-1}A_2)^{-1}P_3J^{-1}B_2]u(t) - g_1(\overline{x_1}(t,r),\overline{x_2}(t,r)) \quad (15)$$

Case 1 :

(The equation(13a) with the lower function)

By using Lemma (2.2) and Laplace transform on (14) , we get

$$(s^{\propto}\underline{X_1}(s,r) - s^{\propto-1}\underline{x}(0,r) = (P_1J^{-1}A_1 - P_1J^{-1}A_2(P_3J^{-1}A_2)^{-1}P_3J^{-1}A_1\underline{X_1}(s,r) + [P_1J^{-1}B_1 - P_1J^{-1}A_2(P_3J^{-1}A_2)^{-1}P_3J^{-1}B_2]U(s) - \mathcal{L}\left(g_1\left(\underline{x_1}(s,r),\underline{x_2}(s,r)\right)\right)$$

Then ,

$$(s^{\propto} - P_1J^{-1}A_1 - P_1J^{-1}A_2(P_3J^{-1}A_2)^{-1}P_3J^{-1}A_1)\underline{X_1}(s,r) = s^{\propto-1}\underline{x}_{1\,0,r} + \left[P_1J^{-1}B_1 - P_1J^{-1}A_2\left(P_3J^{-1}\widehat{A}_2\right)^{-1}P_3J^{-1}B_2\right]U(s,r) - \mathcal{L}\left(g_1\left(\underline{x_1}(s,r),\underline{x_2}(s,r)\right)\right)$$

We get , $\underline{X_1}(s,r)$

$$= (s^{\propto} - P_1J^{-1}A_1 - P_1J^{-1}A_2(P_3J^{-1}A_2)^{-1}P_3J^{-1}A_1)^{-1}\left(\left(s^{\propto-1}\underline{x}_{1\,0,r} + [P_1J^{-1}B_1 - P_1J^{-1}A_2(P_3J^{-1}A_2)^{-1}P_3J^{-1}B_2]U(s,r)\right) - \mathcal{L}\left(g_1\left(\underline{x_1}(s,r),\underline{x_2}(s,r)\right)\right)\right)$$

We obtain ,

$$\underline{X_1}(s,r) = (s^{\propto} - \overline{A}_{11})^{-1}\left(\left(s^{\propto-1}\underline{x}_{1\,0,r} + \widehat{B}_{11}U(s,r)\right) - \mathcal{L}\left(g_1\left(\underline{x_1}(s,r),\underline{x_2}(s,r)\right)\right)\right)$$

Where

$\overline{A}_{11} = P_1J^{-1}A_1 - P_1J^{-1}A_2(P_3J^{-1}A_2)^{-1}P_3J^{-1}A_1$

$,\widehat{B}_{11} = P_1J^{-1}B_1 - P_1J^{-1}A_2(P_3J^{-1}A_2)^{-1}P_3J^{-1}B_2$

Using series expansion to find $(I_{r_1}s^{\propto} - \overline{A}_{11})^{-1}$, one can get

$$(I_{r_1}s^{\propto} - \overline{A}_{11})^{-1} = \sum_{k=0}^{\infty}\left(\frac{\overline{A}_{11}}{s^{\propto}}\right)^k = \sum_{k=0}^{\infty}\overline{A}_{11}{}^k s^{-(k+1)\propto} \quad (16)$$

Hence,

$$\underline{X_1}(s,r) = \sum_{k=0}^{\infty}\overline{A}_{11}{}^k s^{-(k+1)\propto}\begin{pmatrix}\left(s^{\propto-1}\underline{x}_{1\,0,r} + \widehat{B}_{11}U(s,r)\right)\\ -\mathcal{L}\left(g_1\left(\underline{x_1}(s,r),\underline{x_2}(s,r)\right)\right)\end{pmatrix} =$$

$$\left(\left(\sum_{k=0}^{\infty}\overline{A}_{11}{}^k s^{-(k+1)\propto}s^{\propto-1}\underline{x}_{1\,0,r} + \sum_{k=0}^{\infty}\overline{A}_{11}{}^k s^{-(k+1)\propto}\widehat{B}_{11}U(s,r)\right) - \sum_{k=0}^{\infty}\overline{A}_{11}{}^k s^{-(k+1)\propto}\mathcal{L}\left(g_1\left(\underline{x_1}(s,r),\underline{x_2}(s,r)\right)\right)\right)$$

We obtain ,

$$\underline{X_1}(s,r) = \left(\left(\sum_{k=0}^{\infty}\overline{A}_{11}{}^k s^{-k\propto-1}\underline{x}_{1\,0,r} + \sum_{k=0}^{\infty}\overline{A}_{11}{}^k s^{-(k+1)\propto}\widehat{B}_{11}U(s,r)\right) - \sum_{k=0}^{\infty}\overline{A}_{11}{}^k s^{-(k+1)\propto}\mathcal{L}\left(g_1\left(\underline{x_1}(s,r),\underline{x_2}(s,r)\right)\right)\right) \quad (17)$$

By applying the inverse fuzzy Laplace transform for (17), one can get,

$$\underline{x_1}(t,r) = \sum_{k=0}^{\infty}\frac{\overline{A}_{11}{}^k t^{k\propto}}{\Gamma(k\propto+1)}\underline{x}_{1\,0,r} + \int_0^t(t-s)^{\alpha-1}\sum_{k=0}^{\infty}\frac{\overline{A}_{11}{}^k(t-s)^{k\propto}}{\Gamma(k\propto+\alpha)}\widehat{B}_{11}u(s)ds + \int_0^t(t-s)^{\alpha-1}\sum_{k=0}^{\infty}\frac{\overline{A}_{11}{}^k(t-s)^{k\propto}}{\Gamma(k\propto+\alpha)}g_1\left(\underline{x_1}(s,r),\underline{x_2}(s,r)\right)ds$$

$$= E_{\alpha,1}(\overline{A}_{11}t^{\alpha})\underline{x}_{1\,0,r} + \int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(\overline{A}_{11}(t-s)^{\alpha})\widehat{B}_{11}u(s)ds + \int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(\overline{A}_{11}(t-s)^{\alpha})g_1\left(\underline{x_1}(s,r),\underline{x_2}(s,r)\right)ds \quad (18a)$$

Where $E_{\alpha,1}(\overline{A}_{11}t^{\alpha}) = \sum_{k=0}^{\infty} \frac{\overline{A}_{11}{}^{k} t^{k\alpha}}{\Gamma(k\alpha+1)}$

$, E_{\alpha,\alpha}(\overline{A}_{11}(t-s)^{\alpha}) = \sum_{k=0}^{\infty} \frac{\overline{A}_{11}{}^{k}(t-s)^{k\alpha}}{\Gamma(k\alpha+\alpha)}$

Now, using the equation(11b) with the lower function

$\underline{x}_2(t) = -(P_3J^{-1}A_2)^{-1}P_3J^{-1}A_1\underline{x}_1(t)$
$-(P_3J^{-1}A_2)^{-1}P_3J^{-1}B_2u(t)$ (18b)

Case 2 :

(The equation (15) with the upper function)

In similar method on equation (15), we get

$$\overline{x_1}(t,r) = \sum_{k=0}^{\infty} \frac{\overline{A}_{11}{}^{k} t^{k\alpha}}{\Gamma(k\alpha+1)} \overline{x}_{1\,0,r}$$
$$+\int_0^t (t-s)^{\alpha-1} \sum_{k=0}^{\infty} \frac{\overline{A}_{11}{}^{k}(t-s)^{k\alpha}}{\Gamma(k\alpha+\alpha)} \widehat{B}_{11}u(s)ds$$
$$+\int_0^t (t-s)^{\alpha-1} \sum_{k=0}^{\infty} \frac{\overline{A}_{11}{}^{k}(t-s)^{k\alpha}}{\Gamma(k\alpha+\alpha)} g_1(\overline{x_1}(s,r),\overline{x_2}(s,r))ds$$
$$= \sum_{k=0}^{\infty} \frac{\overline{A}_{11}{}^{k} t^{k\alpha}}{\Gamma(k\alpha+1)} \overline{x}_{1\,0,r}$$
$$+\int_0^t (t-s)^{\alpha-1} \sum_{k=0}^{\infty} \frac{\overline{A}_{11}{}^{k}(t-s)^{k\alpha}}{\Gamma(k\alpha+\alpha)} \widehat{B}_{11}u(s)ds$$
$$+\int_0^t (t-s)^{\alpha-1} \sum_{k=0}^{\infty} \frac{\overline{A}_{11}{}^{k}(t-s)^{k\alpha}}{\Gamma(k\alpha+\alpha)} g_1(\overline{x_1}(s,r),\overline{x_2}(s,r))ds$$
$$= E_{\alpha,1}(\overline{A}_{11}t^{\alpha})\overline{x}_{1\,0,r}$$
$$+\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\overline{A}_{11}(t-s)^{\alpha})\, \widehat{B}_{11}u(s)ds$$
$$+\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\overline{A}_{11}(t-s)^{\alpha})\, g_1(\overline{x_1}(s,r),\overline{x_2}(s,r))ds \quad (19a)$$

Where $E_{\alpha,1}(\overline{A}_{11}t^{\alpha}) = \sum_{k=0}^{\infty} \frac{\overline{A}_{11}{}^{k} t^{k\alpha}}{\Gamma(k\alpha+1)}$,

$$E_{\alpha,\alpha}(\overline{A}_{11}(t-s)^{\alpha}) = \sum_{k=0}^{\infty} \frac{\overline{A}_{11}{}^{k}(t-s)^{k\alpha}}{\Gamma(k\alpha+\alpha)}$$

Now, using The equa (11b) with the upper function

$$\overline{x}_2(t) = -(P_3J^{-1}A_2)^{-1}P_3J^{-1}A_1\overline{x}_1(t)$$
$$-(P_3J^{-1}A_2)^{-1}P_3J^{-1}B_2u(t) \qquad (19b)$$

The general solution depended on equa (18) and(19) is

$$\varphi = \begin{bmatrix} \underline{x_1}(t,r), \overline{x_1}(t,r) \\ \underline{x_2}(t,r), \overline{x_2}(t,r) \end{bmatrix}$$
$$= \begin{pmatrix} (E_{\alpha,1}(\overline{A}_{11}t^{\alpha})\underline{x}_{1\,0,r} + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\overline{A}_{11}(t-s)^{\alpha})\,\widehat{B}_{11}u(s)ds \\ + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\overline{A}_{11}(t-s)^{\alpha})\, g_1\left(\underline{x_1}(s,r),\underline{x_2}(s,r)\right)ds,\; E_{\alpha,\alpha}(\overline{A}_{11}t^{\alpha})\overline{x}_{1\,0,r} \\ + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(\overline{A}_{11}(t-s)^{\alpha})\,\widehat{B}_{11}u(s)ds \\ + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\overline{A}_{11}(t-s)^{\alpha})\, g_1(\overline{x_1}(s,r),\overline{x_2}(s,r))ds) \\ \\ (-(P_3J^{-1}A_2)^{-1}P_3J^{-1}A_1\underline{x}_1(t) - (P_3J^{-1}A_2)^{-1}P_3J^{-1}B_2u(t), \\ -(P_3J^{-1}A_2)^{-1}P_3J^{-1}A_1\overline{x}_1(t) - (P_3J^{-1}A_2)^{-1}P_3J^{-1}B_2u(t)) \end{pmatrix}$$

Where $E_{\alpha,\alpha}(\overline{A}_{11}t^{\alpha}) = \sum_{k=0}^{\infty} \frac{\overline{A}_{11}{}^{k} t^{k\alpha}}{\Gamma(k\alpha+1)}$

$, E_{\alpha,\alpha}(\overline{A}_{11}(t-s)^{\alpha}) = \sum_{k=0}^{\infty} \frac{\overline{A}_{11}{}^{k}(t-s)^{k\alpha}}{\Gamma(k\alpha+\alpha)}$

Concluding remark(3.1):

The general solution to the system (9) is given by

$$\varphi = \begin{bmatrix} \underline{x_1}(t,r), \overline{x_1}(t,r) \\ \underline{x_2}(t,r), \overline{x_2}(t,r) \end{bmatrix}$$
$$= \begin{pmatrix} (E_{\alpha,\alpha}(\overline{A}_{11}t^{\alpha})\underline{x}_{1\,0,r} + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\overline{A}_{11}(t-s)^{\alpha})\,\widehat{B}_{11}u(s)ds \\ + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\overline{A}_{11}(t-s)^{\alpha})\, g_1\left(\underline{x_1}(s,r),\underline{x_2}(s,r)\right)ds,\; E_{\alpha,\alpha}(\overline{A}_{11}t^{\alpha})\overline{x}_{1\,0,r} \\ + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\overline{A}_{11}(t-s)^{\alpha})\,\widehat{B}_{11}u(s)ds \\ + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\overline{A}_{11}(t-s)^{\alpha})\, g_1(\overline{x_1}(s,r),\overline{x_2}(s,r))ds) \\ \\ 0 \end{pmatrix}$$

when the coefficient of equa (10b) equal zero .

Illustrative Example (3.1):

Consider the singular fractional order control system as follows:

$E[D^{\alpha}x(t) + g(t,x(t))] = Ax(t) + Bu(t)$,
$t \in [0,G]$ (20)
$x(0) = x_0, 0 < \alpha \le 1$

, such that $E = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} -4 & 0 \\ 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 24 \\ 10 \end{bmatrix}$.

To calculate the matrices $\overline{E}, \overline{A}, \overline{B}$ Firstly , calculate the nonsingular matrix $(Ec - A)$

$\det(Es - A) = \left| \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} s - \begin{bmatrix} -4 & 0 \\ 0 & -2 \end{bmatrix} \right| =$

$\begin{vmatrix} 2s+4 & 0 \\ 0 & 2 \end{vmatrix} = 2s + 4 \neq 0$

suppose c=1, thus ,

$$(Ec - A)^{-1} = \left(\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} -4 & 0 \\ 0 & -2 \end{bmatrix}\right)^{-1}$$

$$= \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

Then $\overline{E} = (Ec - A)^{-1}E$

$$= \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 0 \end{bmatrix},$$

$$\overline{A} = (Ec - A)^{-1}A = \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -4 & 0 \\ 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-2}{3} & 0 \\ 0 & -1 \end{bmatrix}$$

$$\overline{B} = (Ec - A)^{-1}B = \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 24 \\ 10 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

By definition (2.6), calculate the matrix $\overline{E}^D = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$, multiply both sides by drazin inverse of the matric $\overline{E}$. for equation(20) , we get, $\overline{E}^D\overline{E}[D^{\alpha}x(t) + g(t, x(t))]$

$$= \overline{E}^D\overline{A}x(t) + \overline{E}^D\overline{B}u(t)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} D^{\alpha}x_1(t) \\ D^{\alpha}x_2(t) \end{bmatrix} + \begin{bmatrix} g_1(x_1(t), x_2(t))) \\ 0 \end{bmatrix} \right) =$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{-2}{3} & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} u(t)$$

Then,

$$\begin{bmatrix} D^{\alpha}x_1(t) + g_1(x_1(t), x_2(t))) \\ 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 12 \\ 0 \end{bmatrix} u(t)$$

We obtain ,

$$D^{\alpha}x_1(t) + g_1(x_1(t), x_2(t))) = -2x_1(t) + 12u(t) \qquad (21)$$

Now,suppose , $\alpha = 0.5, g_1(x_1(t), x_2(t))) = \lambda x_1(t), t \in R^+ \cup 0, \lambda \in R^+$,

$$D^{0.5}x_1(t) = -2x_1(t) + \lambda x_1(t) + 12u(t) \qquad (22)$$

$x(0) = x_0$, with fuzzy initial condition

$$(\underline{x}(0), \overline{x}(0)) = \left(\begin{bmatrix} \underline{x}_{1,0,r} \\ \underline{x}_{2,0,r} \end{bmatrix}, \begin{bmatrix} \overline{x}_{1,0,r} \\ \overline{x}_{2,0,r} \end{bmatrix} \right)$$

$$= \left(\begin{bmatrix} 1 + r \\ 3 + r \end{bmatrix}, \begin{bmatrix} 2 + r \\ 4 + r \end{bmatrix} \right)$$

Using extended (22) on its lower and upper function

$$\left(D^{0.5}\underline{x}_1(t), D^{0.5}\overline{x}_1(t)\right) = -2\left(\underline{x}_1(t), \overline{x}_1(t)\right) + \lambda\left(\underline{x}_1(t), \overline{x}_1(t)\right) + 12u(t) \qquad (23)$$

Case 1 :

(The equation(23) with the lower function)

$D^{0.5}\underline{x}_1(t) = -2\underline{x}_1(t) + \lambda\underline{x}_1(t) + 12u(t)$

$,\underline{x}_{1,0,r} = (1 + r)$, thus

$D^{0.5}\underline{x}_1(t) = (\lambda - 2)\underline{x}_1(t) + 12u(t)$

$,\underline{x}_{1,0,r} = (1 + r)$,by fuzzy Laplace transform , we get ,

$$s^{0.5}\underline{X_1}(s, r) - s^{0.5-1}\underline{x}_{1,0,r} = (\lambda - 2)\underline{X_1}(s, r) + U(s)$$, follows

$(s^{0.5} - (\lambda - 2))\underline{X_1}(s, r) = s^{-0.5}\underline{x}_{1,0,r} + U(s)$, thus ,

$\underline{X_1}(s, r) = (s^{-0.5} - (\lambda - 2))^{-1}\left(s^{-0.5}\underline{x}_{1,0,r} + U(s)\right)$

, by series expansion to find $(s^{-0.5} - (\lambda - 2))^{-1}$,one can get ,

$(s^{-0.5} - (\lambda - 2))^{-1}$

$= \sum_{k=0}^{\infty}((\lambda - 2)^k s^{-0.5(k+1)})$,

then,

$\underline{X_1}(s, r)$

$$= \sum_{k=0}^{\infty}((\lambda - 2)^k s^{-0.5(k+1)})\left(s^{-0.5}\underline{x}_{1,0,r} + U(s)\right)$$

$$= \sum_{k=0}^{\infty}((\lambda - 2)^k s^{-0.5(k+1)})\, s^{-0.5}\underline{x}_{1,0,r} + \sum_{k=0}^{\infty}((\lambda - 2)^k s^{-0.5(k+1)})\, U(s)$$

$$= \sum_{k=0}^{\infty}((\lambda - 2)^k s^{-0.5(k+1)-0.5})\, \underline{x}_{1,0,r} + \sum_{k=0}^{\infty}((\lambda - 2)^k s^{-0.5(k+1)})\, U(s)$$

$$= \sum_{k=0}^{\infty} ((\lambda-2)^k s^{-0.5k-1})\,(1+r)$$

$$+\sum_{k=0}^{\infty} ((\lambda-2)^k s^{-0.5(k+1)})\,U(s)$$

Using the inverse fuzzy Laplace transform, we get

$\underline{X_1}(t,r)$

$$= (1+r)\sum_{k=0}^{\infty} ((\lambda-2)^k \frac{t^{0.5k}}{\Gamma(0.5k+1})$$

$$+\int_0^t \sum_{k=0}^{\infty} (t-s)^{-0.5}(\lambda - 2)^k \frac{(t-s)^{0.5k}}{\Gamma(0.5k+0.5)} u(s)ds$$

One can get ,

$$\underline{X_1}(t,r) = (1+r)E_{0.5,1}\big((\lambda-2)t\big) + \int_0^t (t-s)^{-0.5}\, E_{0.5,0.5}((\lambda-2)(t-s))u(s)ds \tag{24}$$

Case 2 :

The equation (23) with the upper functions:

In similar method on equation (23), we get $D^{0.5}\bar{x}_1(t) = -2\bar{x}_1(t) + \lambda\bar{x}_1(t) + 12u(t)$, one can get

$$\underline{X_1}(t,r) = (2+r)E_{0.5,1}\big((\lambda-2)t\big) + \int_0^t (t-s)^{-0.5}\, E_{0.5,0.5}((\lambda-2)(t-s))u(s)ds \tag{25}$$

The general solution depended on eq (24) and (25) is

$$\varphi = \begin{bmatrix} \underline{x_1}(t,r)\,,\overline{x_1}(t,r) \\ 0 \end{bmatrix}$$

$$= \begin{pmatrix} ((1+r)E_{0.5,1}\big((\lambda-2)t\big) \\ +\int_0^t (t-s)^{-0.5}\, E_{0.5,0.5}\big((\lambda-2)(t-s)\big)u(s)ds\,,(2+r)E_{0.5,1}\big((\lambda-2)t\big) \\ +\int_0^t (t-s)^{-0.5}\, E_{0.5,0.5}\big((\lambda-2)(t-s)\big)u(s)ds\ ,(2+r)E_{0.5,1}\big((\lambda-2)t\big) \\ +\int_0^t (t-s)^{-0.5}\, E_{0.5,0.5}((\lambda-2)(t-s))u(s)ds) \\ \\ 0 \end{pmatrix}$$

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